One-dimensional heat conduction

	Axial Member	Heat Conduction
Differential Equation	$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = -p_x$ EA= Axial Rigidity p_x= force per unit length	$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = -q_x$ k= Thermal conductivity q_x= Heat flow (source) per unit length. f_x=heat flux per unit area= $q_x A$
Primary variable	u= Displacement in x-direction	T= Temperature
Secondary variable	Internal axial force: $N = EA\frac{du}{dx}$	Heat flow: $q = -k\frac{dT}{dx}$ Positive: Flow into body
Energy	Strain Energy: $U = \frac{1}{2}\int_L EA\left(\frac{du}{dx}\right)^2 dx$	Heat Capacity: $U_T = \frac{1}{2}\int_L k\left(\frac{dT}{dx}\right)^2 dx$
	Work Potential: $W_A = \int_0^L p_x u\,dx + \sum_{q=1}^{m} F_q u(x_q)$	$W_T = \int_0^L q_x T\,dx + \sum_{q=1}^{m} Q_q T(x_q)$
Functional	Potential Energy $\Omega_A = U_A - W_A$	$\Omega_T = U_T - W_T$
Rayleigh-Ritz	$u(x) = \sum_{i=1}^{n} C_i f_i(x)$	$T(x) = \sum_{i=1}^{n} C_i f_i(x)$
Matrix	$K_{jk} = \int_0^L EA\left(\frac{df_j}{dx}\right)\left(\frac{df_k}{dx}\right)dx$	$K_{jk} = \int_0^L k\left(\frac{df_j}{dx}\right)\left(\frac{df_k}{dx}\right)dx$
Right Hand Side Vector	$R_j = \int_0^L p_x f_j\,dx + \sum_{q=1}^{m} F_q f_j(x_q)$	$R_j = \int_0^L q_x f_j\,dx + \sum_{q=1}^{m} Q_q f_j(x_q)$

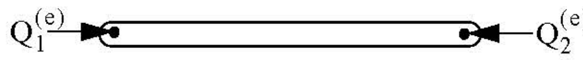
Other Applications:

Flow through pipes; Flow through porous media; Electrostatics

- By non-dimensionalizing the problem a software can be used to solve all the above applications.

Linear Elements:

$$T = T_1^{(e)} L_1(x) + T_2^{(e)} L_2(x)$$

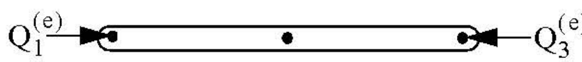


$$[K^{(e)}] = \frac{k^{(e)}}{L^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \{R^{(e)}\} = \frac{q_x^{(e)} L^{(e)}}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^{(e)} \\ Q_2^{(e)} \end{Bmatrix}$$

$$[K^{(e)}] = \frac{k^{(e)} A^{(e)}}{L^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \{R^{(e)}\} = \frac{f_x^{(e)} L^{(e)}}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + A\begin{Bmatrix} Q_1^{(e)} \\ Q_2^{(e)} \end{Bmatrix}$$

Quadratic Elements:

$$T = T_1^{(e)} L_1(x) + T_2^{(e)} L_2(x) + T_3^{(e)} L_3(x)$$



$$[K^{(e)}] = \frac{k^{(e)}}{3L^{(e)}}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \qquad \{R^{(e)}\} = \frac{q_x^{(e)} L^{(e)}}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^{(e)} \\ 0 \\ Q_3^{(e)} \end{Bmatrix}$$

Class Problem 1

Heat q_o is being added at point B at a constant rate as shown in Fig. 1. Using two linear elements determine the temperature at point B and the heat flowing out at A and C in terms of k, L, and q_o for the following two cases: (a) The ends of the bar are maintained at a constant zero temperature. (b) The ends of the bars are maintained at a constant temperature T_o.

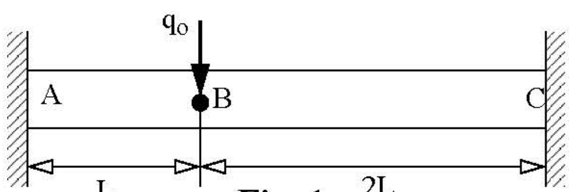


Fig .1

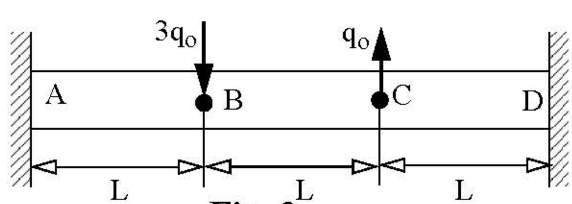


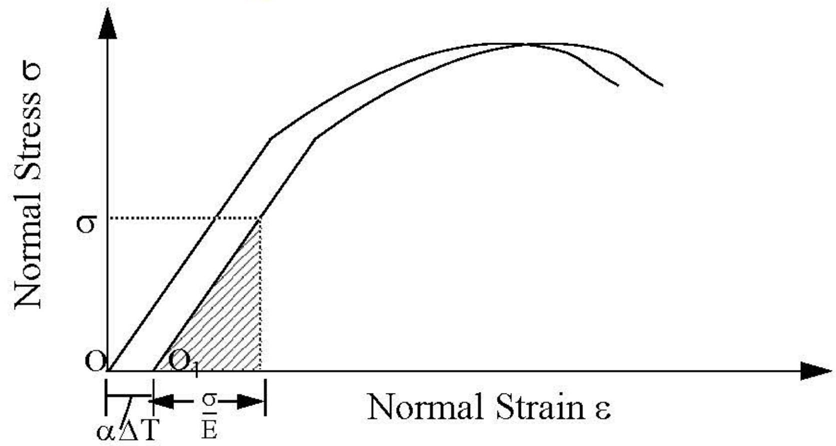
Fig .2

Home Problem 1

Heat $3q_o$ is being added at point B and q_o is being taken out at a constant rate at point C as shown in Fig. 2. Using three linear elements determine the temperature at points B and C and the heat flowing out at A and D in terms of k, L, and q_o for the following two cases: (a) The ends of the bar are maintained at a constant zero temperature. (b) The ends of the bars A and D are maintained at a temperature T_o and $2T_o$, respectively. ANS: $T_B = (5q_oL)/3k$ $\quad T_C = (q_oL)/3k$ $\quad q_A = -(5q_o/3)$

Thermal Stresses

Stress-strain curve with temperature effects



$$\varepsilon = \frac{\sigma}{E} + \alpha \Delta T = \frac{\sigma}{E} + \varepsilon_o$$

where, α = linear coefficient of thermal expansion.

ε_o = Initial strain= Thermal strain

- No thermal stresses are produced in a homogenous, isotropic, unconstrained body due to uniform temperature changes.

Axial Problem

We assume that the thermal problem and stress analysis problem can be solved independently.

$$\sigma_{xx} = E(\varepsilon_{xx} - \varepsilon_o)$$

$$U_{AT} = \int_V \frac{1}{2}\sigma_{xx}(\varepsilon_{xx} - \varepsilon_o)dV = \int_0^L \frac{1}{2}E(\varepsilon_{xx} - \varepsilon_o)^2 A dx \text{ or}$$

$$U_{AT} = \int_0^L \frac{1}{2}EA\varepsilon_{xx}^2 dx - \int_0^L EA\varepsilon_{xx}\varepsilon_o dx + \int_0^L \frac{1}{2}EA\varepsilon_o^2 dx = \int_0^L \frac{1}{2}EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L EA\varepsilon_o\left(\frac{du}{dx}\right)dx + U_o$$

$$W_A = \int_0^L p_x(x)u(x)dx + \sum_{q=1}^{m} F_q u(x_q)$$

$$\Omega_A = U_{AT} - W_A = \int_0^L \frac{1}{2}EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L EA\varepsilon_o\left(\frac{du}{dx}\right)dx + U_o - \left[\int_0^L p_x(x)u(x)dx + \sum_{q=1}^{m} F_q u(x_q)\right]$$

$$\Omega_A = \int_0^L \frac{1}{2}EA\left(\frac{du}{dx}\right)^2 dx + U_o - \left[\int_0^L p_x(x)u(x)dx + \sum_{q=1}^{m} F_q u(x_q) + \int_0^L EA\varepsilon_o\left(\frac{du}{dx}\right)dx\right] = U_A - W_{AT}$$

$$W_{AT} = \int_0^L p_x(x)u(x)dx + \sum_{q=1}^{m} F_q u(x_q) + \int_0^L EA\varepsilon_o\left(\frac{du}{dx}\right)dx$$

$$I = \int_0^L EA\varepsilon_o\left(\frac{du}{dx}\right)dx = EA\varepsilon_o u(x)\Big|_0^L - \int_0^L EA\left(\frac{d\varepsilon_o}{dx}\right)u\,dx = EA\alpha\Delta T u(x)\Big|_0^L - \int_0^L EA\alpha\left(\frac{d}{dx}\Delta T\right)u\,dx$$

$$I = EA\alpha\Delta T_L u(L) - EA\alpha\Delta T_0 u(0) + \int_0^L\left(\frac{EA\alpha}{k}q_x\right)u\,dx, \text{ where } q_x = -k\frac{dT}{dx}$$

$$W_{AT} = \int_0^L [p_x(x) + p_{xT}(x)]u(x)dx + \sum_{q=1}^{m} F_q u(x_q) + F_{T2}u(L) + F_{T1}u(0)$$

where, $\begin{Bmatrix} F_{T1} \\ F_{T2} \end{Bmatrix} = \begin{Bmatrix} -EA\alpha\Delta T_1 \\ EA\alpha\Delta T_2 \end{Bmatrix}$ $\qquad p_{xT} = \frac{EA\alpha}{k}q_x$

Special case: Constant Temperature Change

$\Delta T = \text{Constant} \qquad q_x = 0$

- Thermal loads are added only at the element ends.

Linear Element:

$$[K^{(e)}] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \{R^{(e)}\} = \frac{p_o L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \frac{p_{To} L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} F_1^{(e)} \\ F_2^{(e)} \end{Bmatrix} + \begin{Bmatrix} F_{T1}^{(e)} \\ F_{T2}^{(e)} \end{Bmatrix}$$

Quadratic Element:

$$[K^{(e)}] = \frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \qquad \{R\} = \frac{p_o L}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \frac{p_{To} L}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} F_1^{(e)} \\ 0 \\ F_3^{(e)} \end{Bmatrix} + \begin{Bmatrix} F_{T1}^{(e)} \\ 0 \\ F_{T3}^{(e)} \end{Bmatrix}$$

Class Problem 2

Heat q_o is being added at point B at a constant rate as shown in Fig. 3. Using two quadratic elements determine the displacement of node B, reaction force at A and the axial stress just before B in terms of E, A, α, k, L, and q_o. Assume that the entire bar was at zero temperature and the ends of the bars are maintained at zero temperature.

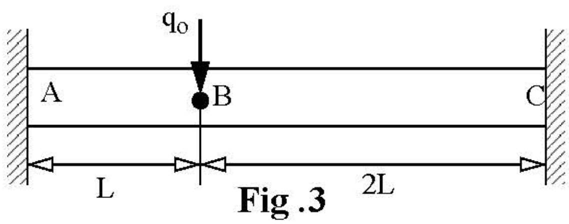


Fig .3

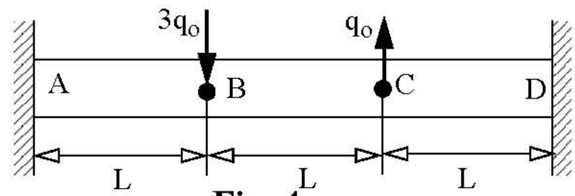


Fig .4

Home Problem 2

Heat $3q_o$ is being added at point B and q_o is being taken out at a constant rate at point C as shown in Fig. 4. Using three linear elements determine the displacements of points B and C, the reaction force at A, and the axial stress just before B in terms of E, A, α, k, L, and q_o. Assume that the entire bar was at zero temperature and the ends of the bars are maintained at zero temperature.

ANS: $u_B = \dfrac{5\alpha q_o L^2}{18k}$ $\quad u_C = \dfrac{7\alpha q_o L^2}{18k}$ $\quad R_A = \dfrac{5EA\alpha q_o L}{9k}$

2-D Steady State Thermal Analysis

Differential Equation: $k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) = -q_v$ **or** $k\nabla^2 T = -q_v$

Functional (Stored Heat):

$$U_T = \iint_A k\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right] t\,dx\,dy$$

where t= thickness of the body.

$$U_T = \iint_A \begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{Bmatrix}^T [k] \begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{Bmatrix} t\,dx\,dy$$

Isotropic material: $[k] = k$ **---scaler quantity.**

Element Approximation:

$$T(x) = \sum_{i=1}^{n} T_i^{(e)} f_i(x,y) = \begin{bmatrix} f_1 & f_2 & \dots & \dots & f_n \end{bmatrix} \begin{Bmatrix} T_1^{(e)} \\ T_2^{(e)} \\ \\ T_n^{(e)} \end{Bmatrix}$$

$f_i(x,y)$ are Lagrange Polynomials.

$$\begin{Bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} & \frac{\partial f_3}{\partial x} & \dots \dots & \frac{\partial f_n}{\partial x} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} & \frac{\partial f_3}{\partial y} & \dots \dots & \frac{\partial f_n}{\partial y} \end{bmatrix} \begin{Bmatrix} T_1^{(e)} \\ T_2^{(e)} \\ \\ T_n^{(e)} \end{Bmatrix} = [B^{(e)}]\{d^{(e)}\}$$

$$U_T = \iint_A \frac{1}{2}\{d^{(e)}\}^T [B^{(e)}]^T [k^{(e)}][B^{(e)}]\{d^{(e)}\} t\,dx\,dy = \frac{1}{2}\{d^{(e)}\}^T [K_T^{(e)}]\{d^{(e)}\}$$

Element Conductivity Matrix

$$[K_T^{(e)}] = \iint_A [B^{(e)}]^T [k^{(e)}][B^{(e)}] t\,dx\,dy$$

Heat Conduction Boundary Conditions

$$-k\frac{\partial T}{\partial n} = -k\left(\frac{\partial T}{\partial x}n_x + \frac{\partial T}{\partial y}n_y\right) = q_n$$

where, n= direction of the unit normal to the boundary.

n_x, n_y = direction cosines of the unit normal to the boundary.

q_n = specified heat flow in the n-direction on the boundary.

Right Hand Side Vector:

$$\{R^{(e)}\} = \iint_{A^{(e)}} q_v \begin{Bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ f_n \end{Bmatrix} t\,dx\,dy + \int_{\Gamma^{(e)}} q_n \begin{Bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ f_n \end{Bmatrix} t\,ds$$

where, $\Gamma^{(e)}$ = the boundary of the element.

s= tangential coordinate along the element boundary.

Convection Boundary Conditions

$$-k\frac{\partial T}{\partial n} = h(T_f - T)$$

where, h = convection heat transfer coefficients

T_f = Temperature of the surrounding fluid

- h depends upon many factors: velocity of fluid, viscosity of fluid, density of fluid, and other properties of fluid. It also depends upon the surface roughness and surface geometry.

Addition to Element Matrix: $K_{ij}^{(e)} = \int_{\Gamma^{(e)}} h^{(e)} f_i f_j t\,ds$

Addition to Element Right Hand Side Vector: $R_i^{(e)} = \int_{\Gamma^{(e)}} h^{(e)} f_i T_f t\,ds$

- If $\Gamma^{(e)}$ is an element boundary in the interior, then there is no convection there and hence no addition to the matrix or the element RHS vector.

Radiation Boundary Condition:

Heat radiated is proportional to the difference in the fourth power of temperature between the radiating bodies.

$$-k\frac{\partial T}{\partial n} = B_C(T_r^4 - T^4)$$

where, B_C is the proportionality constant.

T_r is the temperature of the other radiating body.

Temperatures T_r and T are in absolute degrees i.e., ^oK.

- For two infinite parallel black bodies (planes) it is called the Boltzmann constant. For regular bodies B_C depends upon the emissivity of the bodies, the geometry and other factors.
- Radiation boundary conditions lead to non-linear thermal problem.

A general approach is:

$$-k\frac{\partial T}{\partial n} = B_C(T_r^2 + T^2)(T_r + T)(T_r - T) = h_r(T_r - T)$$

In the iteration process, at each step treat the radiation boundary condition like a convection term with coefficient dependent upon the temperature at a particular step.