REVIEW OF MECHANICS OF MATERIALS

1.0 Stress at a Point

$$\sigma_{ij} = \lim_{\Delta A_i \to 0}\left(\frac{\Delta F_j}{\Delta A_i}\right)$$

direction of outward normal to the imaginary cut surface. (i)

direction of the internal force. (j)

1. Stress is an internal quantity.
2. Stress has units of force per unit area.
3. Stress at a point needs a magnitude and two directions to specify it (i.e. stress is a second-order tensor).
4. The sign of a stress component is determined from the direction of the internal force and the direction of the outward normal to the imaginary cut surface.
5. Shear stress components are symmetric

2.0 Strain

Measure of relative movement of two points on the body. (deformation)

1. Elongations are positive normal strains. Decrease from right angle results in positive shear strains.
2. Small strain ($\varepsilon < 0.01$) can be calculated using just the deformation in the original direction of the line.
3. Small strain results in a linear theory

Engineering Strain $\varepsilon_{xx} = \frac{\partial u}{\partial x}$ $\quad \varepsilon_{yy} = \frac{\partial v}{\partial y}$ $\quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$ $\quad \gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ $\quad \gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$ $\quad \gamma_{zx} = \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$

4. tensor normal strains = engineering normal strains and tensor shear strains = (engineering shear strains)/ 2

3.0 Generalized Hooke's Law

Assuming no temperature change, we have the following:

For isotropic materials

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}(\sigma_{yy} + \sigma_{zz}) + \alpha\Delta T \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{zz}) + \alpha\Delta T \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) + \alpha\Delta T \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

$$G = \frac{E}{2(1+\nu)}$$

$$\text{Plane Stress} \longrightarrow \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Generalized Hooke's Law}} \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \end{bmatrix}$$

$$\text{Plane Strain} \longrightarrow \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Generalized Hooke's Law}} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \end{bmatrix}$$

4.0 Stress Transformation

- Stress transformation equations relate stresses *at a point* in different coordinate systems.

$$\sigma_{nn} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \qquad \tau_{nt} = -\sigma_{xx}\cos\theta\sin\theta + \sigma_{yy}\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

where, σ_{xx}, σ_{yy}, and τ_{xy} are the stresses in x-y-z coordinate system, σ_{nn}, σ_{tt}, and τ_{nt}, are the stresses in n-t-z coordinate system, θ is measured from the x-axis in the counter-clockwise direction to the n-direction.

- The value of stresses on a plane through a point are unique and depend upon the orientation of the plane only and not how its orientation is described or measured.
- Planes on which the shear stresses are zero are called the *principal planes*.
- Principal planes are orthogonal.
- The normal direction to the principal planes is referred to as the principal direction or the *principal axis*.
- The angles the principal axis makes with the global coordinate system are called the *principal angles*.
- Normal stress on a principal plane is called *principal stress*.

- The greatest principal stress is called *principal stress one.*
- Principal stresses are the maximum and minimum normal stresses at a point.
- The maximum shear stress on a plane that can be obtained by rotating about the z axis is called the *in-plane maximum shear stress.*
- The maximum shear stress at a point is the *absolute maximum shear stress* that is on any plane passing through the point.
- Maximum in-plane shear stress exists on a two plane which are at 45^{o} to the principal planes.

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_{xx} - \sigma_{yy})} \qquad \sigma_{1,2} = \frac{(\sigma_{xx} + \sigma_{yy})}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \qquad |\tau_p| = \left|\frac{\sigma_1 - \sigma_2}{2}\right|$$

where, θ_p is the angle to either principal plane one *or* two, σ_1 and σ_2 are the principal stresses, τ_p is the in-plane maximum shear stress.

- $\sigma_3 = \sigma_{zz} = \begin{cases} 0 & \text{Plane Stress} \\ \nu(\sigma_{xx} + \sigma_{yy}) & \text{Plane Strain} \end{cases} \qquad \tau_{max} = \left|\max\left(\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2}\right)\right|$
- At a point there are always three principal stresses.

5.0 Failure Theories

Maximum shear stress theory: $|\max(\sigma_1 - \sigma_2, \sigma_2 - \sigma_3, \sigma_3 - \sigma_1)| \le \sigma_{yield}$ ---Ductile materials

Maximum octahedral shear stress: $\sigma_{von} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \le \sigma_{yield}$ ---Ductile materials

σ_{von} is called von-Mises stress.

Maximum normal stress theory $|\max(\sigma_1, \sigma_2, \sigma_3)| \le \sigma_{ult}$ ---Brittle materials

Modified Mohr's theory: $\left|\frac{\sigma_2}{\sigma_C} - \frac{\sigma_1}{\sigma_T}\right| \le 1$ ---Brittle materials with different tensile and compressive strength.

6.0 Strain Transformation

- Strain transformation equations relate strains *at a point* in different coordinate systems.

$$\varepsilon_{nn} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta \qquad \gamma_{nt} = -2\varepsilon_{xx}\sin\theta\cos\theta + 2\varepsilon_{yy}\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{(\varepsilon_{xx} - \varepsilon_{yy})} \qquad \varepsilon_{1,2} = \frac{(\varepsilon_{xx} + \varepsilon_{yy})}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad \left|\frac{\gamma_p}{2}\right| = \left|\frac{\varepsilon_1 - \varepsilon_2}{2}\right|$$

where, θ_p is the angle to either principal plane one *or* two, ε_1 and ε_2 are the principal stresses, γ_p is the in-plane maximum shear stress.

$$\varepsilon_3 = \begin{cases} 0 & \text{Plane Strain} \\ -\left(\frac{\nu}{1-\nu}\right)(\varepsilon_{xx} + \varepsilon_{yy}) & \text{Plane Stress} \end{cases} \qquad \frac{\gamma_{max}}{2} = \left|\max\left(\frac{\varepsilon_1 - \varepsilon_2}{2}, \frac{\varepsilon_2 - \varepsilon_3}{2}, \frac{\varepsilon_3 - \varepsilon_1}{2}\right)\right|$$

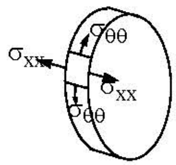
- The principal directions for stresses and strains is same for isotropic materials.

- Generalized Hooke's Law in principal coordinates:

$$\varepsilon_1 = [\sigma_1 - \nu(\sigma_2 + \sigma_3)]/E$$
$$\varepsilon_2 = [\sigma_2 - \nu(\sigma_3 + \sigma_1)]/E$$
$$\varepsilon_3 = [\sigma_3 - \nu(\sigma_1 + \sigma_2)]/E$$

7.0 Pressure Vessels

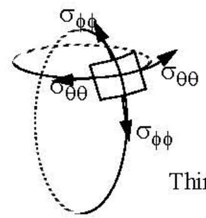
Stress Approximation
1. Free Surface
2. Thin Bodies.
3. Axi-symmetric Bodies.



Hoop stress: $\sigma_{\theta\theta} = \frac{pR}{t}$

Axial stress: $\sigma_{xx} = \frac{pR}{2t}$

Thin cylindrical pressure vessels



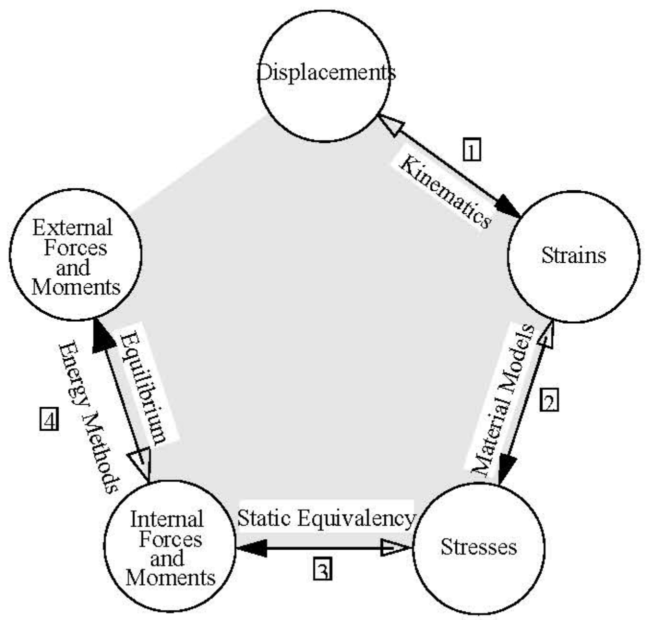
$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \sigma = \frac{pR}{2t}$$

Thin spherical pressure vessels

$$\tau_{xs} = \frac{T}{2tA_E}$$

Torsion of thin tubes

8.0 Logic in Structural Mechanics



Structural Analysis

	Axial (Rods)	Torsion (Shafts)	Symmetric Bending (Beams)	Unsymmetric Bending
Displacements	$u(x, y, z) = u(x)$	$\phi(x, y, z) = \phi(x)$	$u(x, y, z) = -y\frac{dv}{dx}$ $v = v(x)$ $w = 0$	$u(x, y, z) = -y\frac{dv}{dx} - z\frac{dw}{dx}$ $v = v(x)$ $w = w(x)$
Strains	$\varepsilon_{xx} = \frac{du}{dx}$	$\gamma_{x\theta} = \rho\frac{d\phi}{dx}$	$\varepsilon_{xx} = -y\frac{d^2v}{dx^2}$	$\varepsilon_{xx} = -y\frac{d^2v}{dx^2} - z\frac{d^2w}{dx^2}$
Stresses	$\sigma_{xx} = E\varepsilon_{xx} = E\frac{du}{dx}$	$\tau_{x\theta} = G\gamma_{x\theta} = G\rho\frac{d\phi}{dx}$	$\sigma_{xx} = E\varepsilon_{xx} = -Ey\frac{d^2v}{dx^2}$ $\tau_{xy} \neq 0 \ll \sigma_{xx}$	$\sigma_{xx} = -Ey\frac{d^2v}{dx^2} - Ez\frac{d^2w}{dx^2}$ $\tau_{xy} \neq 0 \ll \sigma_{xx}$ $\tau_{xz} \neq 0 \ll \sigma_{xx}$
Internal Forces & Moments	$N = \int_A \sigma_{xx} dA$	$T = \int_A \rho\tau_{x\theta} dA$	$N = \int_A \sigma_{xx} dA = 0$ $M_z = -\int_A y\sigma_{xx} dA$ $V_y = \int_A \tau_{xy} dA$	$N = \int_A \sigma_{xx} dA = 0$ $M_z = -\int_A y\sigma_{xx} dA$ $M_y = -\int_A z\sigma_{xx} dA$ $V_y = \int_A \tau_{xy} dA$ $V_z = \int_A \tau_{xz} dA$
Homogeneous Cross-section	$\sigma_{xx} = \frac{N}{A}$	$\tau_{x\theta} = \frac{T\rho}{J}$	$\sigma_{xx} = -\left(\frac{M_z y}{I_{zz}}\right)$ $q = \tau_{xs}t = -\left(\frac{V_y Q_z}{I_{zz}}\right)$	$\sigma_{xx} = -\left(\frac{I_{yy}M_z - I_{yz}M_y}{I_{yy}I_{zz} - I_{yz}^2}\right)y - \left(\frac{I_{zz}M_y - I_{yz}M_z}{I_{yy}I_{zz} - I_{yz}^2}\right)z$ $q = \tau_{xs}t = -\left(\frac{I_{yy}Q_z - I_{yz}Q_y}{I_{yy}I_{zz} - I_{yz}^2}\right)V_y - \left(\frac{I_{zz}Q_y - I_{yz}Q_z}{I_{yy}I_{zz} - I_{yz}^2}\right)V_z$
	$\frac{du}{dx} = \frac{N}{EA}$ $u_2 - u_1 = \frac{N(x_2 - x_1)}{EA}$	$\frac{d\phi}{dx} = \frac{T}{GJ}$ $\phi_2 - \phi_1 = \frac{T(x_2 - x_1)}{GJ}$	$\frac{d^2v}{dx^2} = \frac{M_z}{EI_{zz}}$ $v = \int\left[\int\frac{M_z}{EI}dx\right]dx + C_1x + C_2$	$\frac{d^2v}{dx^2} = \frac{1}{E}\left(\frac{I_{yy}M_z - I_{yz}M_y}{I_{yy}I_{zz} - I_{yz}^2}\right)$ $\frac{d^2w}{dx^2} = \frac{1}{E}\left(\frac{I_{zz}M_y - I_{yz}M_z}{I_{yy}I_{zz} - I_{yz}^2}\right)$
Composite Cross-section	$(\sigma_{xx})_i = \frac{NE_i}{\sum^{n} E_jA_j}$	$(\tau_{x\theta})_i = \frac{G_i\rho T}{\left[\sum^{n} G_jJ_j\right]}$	$(\sigma_{xx})_i = -\left[\frac{E_i y M_z}{\sum_{i=1}^{n} E_j(I_{zz})_j}\right]$ $q = \tau_{xs}t = -\left\{\frac{Q_{comp}V_y}{\left[\sum^{n} E_j(I_{zz})_j\right]}\right\}$	
	$u_2 - u_1 = \frac{N(x_2 - x_1)}{\sum E_iA_i}$	$\phi_2 - \phi_1 = \frac{T(x_2 - x_1)}{[\sum G_iJ_i]}$	$v = \int\left[\int\frac{M_z}{\sum E_j(I_{zz})_j}dx\right]dx + C_1x + C_2$	
	$\frac{dN}{dx} = -p_x(x)$	$\frac{dT}{dx} = -t(x)$	$\frac{dV_y}{dx} = -p_y(x)$ $\frac{dM_z}{dx} = -V_y$	$\frac{dV_y}{dx} = -p_y(x)$ $\frac{dM_z}{dx} = -V_y$ $\frac{dV_z}{dx} = -p_z(x)$ $\frac{dM_y}{dx} = -V_z$
	$\frac{d}{dx}\left(EA\frac{du_o}{dx}\right) = -p_x(x)$	$\frac{d}{dx}\left(GJ\frac{d\phi}{dx}\right) = -t(x)$	$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = p_y(x)$	

9.0 Buckling:

Bending due to *compressive* axial forces is called buckling.
It is sudden and catastrophic.
Buckling occurs about the axis of *minimum* area moment of inertia.

Euler Buckling Load P_{cr} can be calculated from: $P_{cr} = \dfrac{\pi^2 EI}{L^2}$

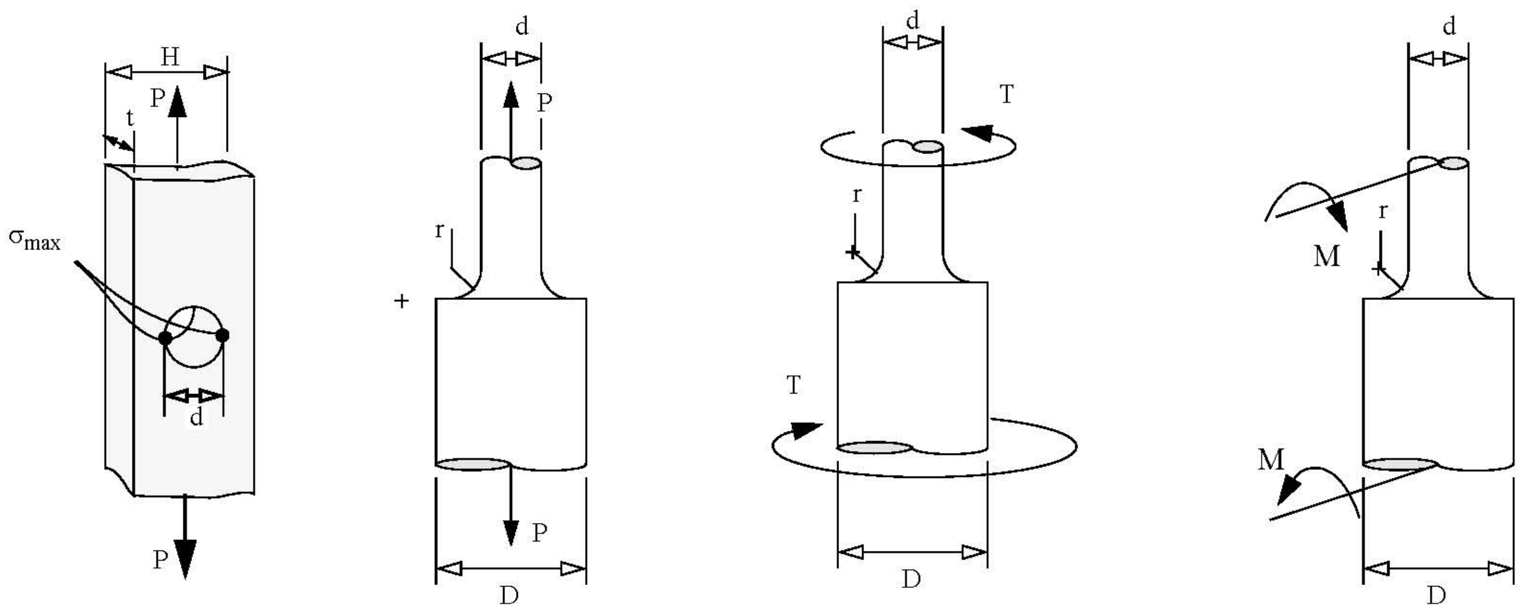
Slenderness ratio is defined as L/r where L is length of column and r is radius of gyration.

10.0 Stress Concentration

Sudden changes in geometry, loading, and material properties cause a concentration of stresses. The impact of the stress raisers (causing the stress concentration) dies out rapidly with distance as per Saint Venant's principle.

Stress concentration factor: $K_{conc} = \dfrac{\text{Maximum Stress}}{\text{Nominal Stress}}$.

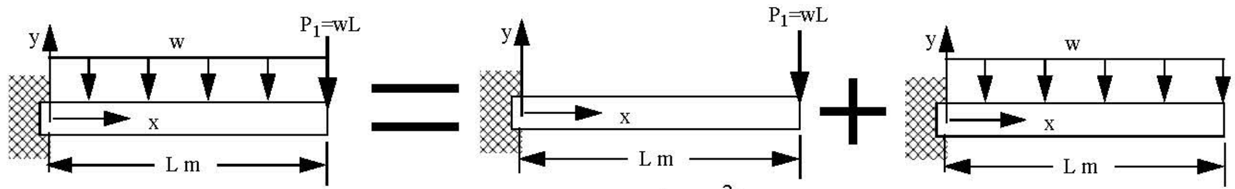
The formulas of mechanics of materials give us the nominal stress. Use charts in handbooks to obtain K_{conc} and find the predicted maximum stress.



11.0 Superposition

Applicable only to linear systems.

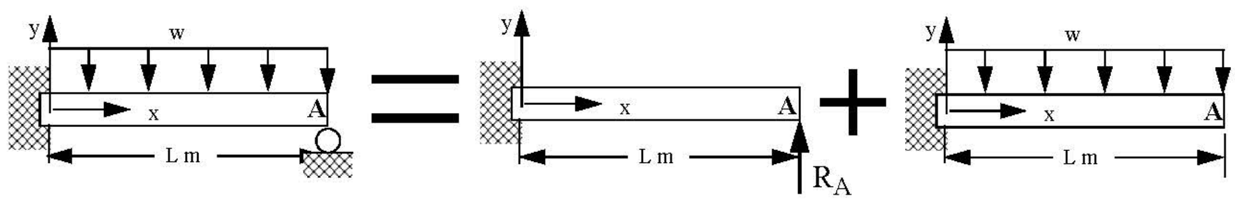
Example 1:



$$v_{max} = -\left(\frac{P_1 L^3}{3EI}\right) \qquad v_{max} = -\left(\frac{wL^4}{8EI}\right)$$

$$v_{max} = \left(-\frac{(wL)L^3}{3EI} - \frac{wL^4}{8EI}\right) = -\left(\frac{11wL^4}{24EI}\right)$$

Example 2:



$$v_A = \left(\frac{R_A L^3}{3EI} - \frac{wL^4}{8EI}\right) = 0 \qquad \text{or} \qquad R_A = \frac{3wL}{8EI}$$