Energy Methods

- Minimum-energy principles are an alternative to statement of equilibrium equations.

The learning objectives in this chapter is:

- Understand the perspective and concepts in energy methods.

**Strain Energy**

- The energy stored in a body due to deformation is called the *strain energy*.
- The strain energy per unit volume is called the *strain energy density* and is the area underneath the stress-strain curve up to the point of deformation.

\[
\bar{U}_o = \text{Complimentary strain energy density}
\]

\[
dU_o = \varepsilon \, d\sigma \quad \text{and} \quad \sigma \, d\varepsilon
\]

Strain Energy:

\[
U = \int_{V}^{\varepsilon} U_o \, dV
\]

Strain Energy Density:

\[
U_o = \int_{0}^{\sigma} \sigma \, d\varepsilon
\]

Units:

- \( \text{N-m} / \text{m}^3 \), Joules / \( \text{m}^3 \), \( \text{in-lbs} / \text{in}^3 \), or \( \text{ft-lb/ft}^3 \)

Complimentary Strain Energy Density:

\[
\bar{U}_o = \int_{0}^{\varepsilon} \varepsilon \, d\sigma
\]
Linear Strain Energy Density

Uniaxial tension test:

\[ U_o = \int_0^\varepsilon \sigma \, d\varepsilon = \int_0^\varepsilon (E\varepsilon) \, d\varepsilon = \frac{E\varepsilon^2}{2} = \frac{1}{2} \sigma \varepsilon \]

\[ U_o = \frac{1}{2} \tau \gamma \]

- Strain energy and strain energy density is a scalar quantity.

\[ U_o = \frac{1}{2} [\sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + \sigma_{zz} \varepsilon_{zz} + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx}] \]

1-D Structural Elements

Axial strain energy
- All stress components except \( \sigma_{xx} \) are zero.

\[ \sigma_{xx} = E\varepsilon_{xx} \quad \varepsilon_{xx} = \frac{du}{dx}(x) \]

\[ U_A = \int_0^L \int_{-A}^A \int_v \frac{1}{2} E v_x^2 \, dV = \int_0^L \left[ \int_{-A}^A \left( \frac{du}{dx} \right)^2 \, dA \right] \, dx \]

\[ U_A = \int U_a \, dx \quad U_a = \frac{1}{2} EA \left( \frac{du}{dx} \right)^2 \]
- \( U_a \) is the strain energy per unit length.

\[ \overline{U}_A = \int_0^L \overline{U}_a \, dx \quad \overline{U}_a = \frac{1}{2} \frac{N^2}{EA} \]
Torsional strain energy

- All stress components except $\tau_{x\theta}$ in polar coordinate are zero

$$\tau_{x\theta} = G\gamma_{x\theta}, \quad \gamma_{x\theta} = \rho\frac{d\phi}{dx}(x)$$

$$U_T = \int \frac{1}{2} G \gamma_{x\theta}^2 dV = \int \left[ \int \frac{1}{2} G \left( \rho \frac{d\phi}{dx} \right)^2 dA \right] dx = \int \left[ \int \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 \int G \rho^2 dA \right] dx$$

$$U_T = \int U_T \ dx \quad U_T = \frac{1}{2} G J \left( \frac{d\phi}{dx} \right)^2$$

- $U_T$ is the strain energy per unit length.

$$\bar{U}_T = \int \frac{\bar{U}_T}{L} \ dx \quad \bar{U}_T = \frac{1}{2} \frac{T^2}{GJ}$$

Strain energy in symmetric bending about z-axis

There are two non-zero stress components, $\sigma_{xx}$ and $\tau_{xy}$.

$$\sigma_{xx} = E \varepsilon_{xx}, \quad \varepsilon_{xx} = -y \frac{d^2 v}{dx^2}$$

$$U_B = \int \frac{1}{2} E \varepsilon_{xx}^2 dV = \int \left[ \int \frac{1}{2} E \left( y \frac{d^2 v}{dx^2} \right)^2 dA \right] dx = \int \left[ \int \frac{1}{2} \left( \frac{d^2 v}{dx^2} \right)^2 \int E y^2 dA \right] dx$$

$$U_B = \int U_B \ dx \quad U_B = \frac{1}{2} E I_{zz} \left( \frac{d^2 v}{dx^2} \right)^2$$

- where $U_B$ is the bending strain energy per unit length.

$$\bar{U}_B = \int \frac{\bar{U}_B}{L} \ dx \quad \bar{U}_B = \frac{1}{2} \frac{M_z^2}{2EI_{zz}}$$

The strain energy due to shear in bending is:

$$U_s = \int \frac{1}{2} \tau_{xy} \gamma_{xy} \ dV = \int \frac{1}{2} \frac{\gamma_{xy}^2}{E} \ dV$$

As $\tau_{max} \ll \sigma_{max}$

$$U_s \ll U_B$$
<table>
<thead>
<tr>
<th></th>
<th>Strain energy density per unit length</th>
<th>Complimentary strain energy density per unit length</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Axial</strong></td>
<td>( U_a = \frac{1}{2} EA \left( \frac{dh}{dx} \right)^2 )</td>
<td>( \bar{U}_a = \frac{1}{2} \frac{N^2}{EA} )</td>
</tr>
<tr>
<td><strong>Torsion of circular shafts</strong></td>
<td>( U_t = \frac{1}{2} GJ \left( \frac{d\phi}{dx} \right)^2 )</td>
<td>( \bar{U}_t = \frac{1}{2} \frac{T^2}{GJ} )</td>
</tr>
<tr>
<td><strong>Symmetric bending of beams</strong></td>
<td>( U_b = \frac{1}{2} EI_{zz} \left( \frac{d^2 v}{dx^2} \right)^2 )</td>
<td>( \bar{U}<em>b = \frac{1}{2} \frac{M_z^2}{EI</em>{zz}} )</td>
</tr>
</tbody>
</table>
Work

- If a force moves through a distance, then work has been done by the force.
  \[ dW = F\, du \]
- Work done by a force is conservative if it is path independent.
- Non-linear systems and non-conservative systems are two independent description of a system.

LOADING MODE

\begin{align*}
\delta W &= P\delta u_L \\
\delta W &= \int_0^L p(x)\delta u(x)\, dx \\
\delta W &= T\delta \phi_L \\
\delta W &= \int_0^L t(x)\delta \phi(x)\, dx \\
\delta W &= P\delta v_L \\
\delta W &= M\delta \theta_L \\
\delta W &= \int_0^L p(x)\delta v(x)\, dx
\end{align*}

- Any variable that can be used for describing deformation is called the generalized displacement.
- Any variable that can be used for describing the cause that produces deformation is called the generalized force.
Virtual Work

- Virtual work methods are applicable to linear and non-linear systems, to conservative as well as non-conservative systems.

The principle of virtual work:

*The total virtual work done on a body at equilibrium is zero.*

\[ \delta W = 0 \]

- Symbol \( \delta \) will be used to designate a virtual quantity

\[ \delta W_{\text{ext}} = \delta W_{\text{int}} \]

**Types of boundary conditions**

- **Kinematic variable (Primary variable)**
  - \( u \)
  - \( \phi \)
  - \( v \)
  - \( \theta = \frac{dv}{dx} \)

- **Statical variable (Secondary variable)**
  - \( N \)
  - \( T \)
  - \( V_y \)
  - \( M_z \)

**Geometric boundary conditions (Kinematic boundary conditions)**

(Essential boundary conditions):

Condition specified on kinematic (primary) variable at the boundary.

**Statical boundary conditions**

(Natural boundary conditions)

Condition specified on statical (secondary) variable at the boundary.
Kinematically admissible functions

- Functions that are continuous and satisfies all the kinematic boundary conditions are called *kinematically admissible functions*.
- Actual displacement solution is always a kinematically admissible function.
- Kinematically admissible functions are not required to correspond to solutions that satisfy equilibrium equations.

Statically admissible functions

- Functions that satisfy satisfies all the static boundary conditions, satisfy equilibrium equations at all points, and are continuous at all points except where a concentrated force or moment is applied are called *statically admissible functions*.
- Actual internal forces and moments are always statically admissible.
- Statically admissible functions are not required to correspond to solutions that satisfy compatibility equations.

7.3 Determine a class of kinematically admissible displacement functions for the beam shown in Fig. P7.3.

![Fig. P7.3](image)

7.4 For the beam and loading shown in Fig. P7.3 determine a statically admissible bending moment.
Virtual displacement method

- The virtual displacement is an infinitesimal imaginary kinematically admissible displacement field imposed on a body.

- Of all the virtual displacements the one that satisfies the virtual work principle is the actual displacement field.

Virtual Force Method

- The virtual force is an infinitesimal imaginary statically admissible force field imposed on a body.
- Of all the virtual force fields the one that satisfies the virtual work principle is the actual force field.
7.7  The roller at P shown in Fig. P7.7 slides in the slot due to the force \( F = 20 \text{kN} \). Both bars have a cross-sectional area of \( A = 100 \text{ mm}^2 \) and a modulus of elasticity \( E = 200 \text{ GPa} \). Bar AP and BP have lengths of \( L_{\text{AP}} = 200 \text{ mm} \) and \( L_{\text{BP}} = 250 \text{ mm} \) respectively. Determine the axial stress in the member AP by virtual displacement method.

![Fig. P7.7](image)

7.8  A force \( F = 20 \text{kN} \) is applied pin shown in Fig. P7.8. Both bars have a cross-sectional area of \( A = 100 \text{ mm}^2 \) and a modulus of elasticity \( E = 200 \text{ GPa} \). Bar AP and BP have lengths of \( L_{\text{AP}} = 200 \text{ mm} \) and \( L_{\text{BP}} = 250 \text{ mm} \) respectively. Using virtual force method determine the movement of pin in the direction of force F.

![Fig. P7.8](image)