Polar Coordinates

The small strain-displacement equations in polar coordinates are:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \qquad \gamma_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u_r}{\partial z} \qquad \gamma_{z\theta} = \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v_\theta}{\partial z}$$

The Generalized Hooke's Law can be written as:

$$\sigma_{rr} = \frac{2G}{(1-2\nu)}[(1-\nu)\varepsilon_{rr} + \nu\varepsilon_{\theta\theta} + \nu\varepsilon_{zz}] \qquad \gamma_{r\theta} = \frac{\tau_{r\theta}}{G}$$

$$\sigma_{\theta\theta} = \frac{2G}{(1-2\nu)}[(1-\nu)\varepsilon_{\theta\theta} + \nu\varepsilon_{rr} + \nu\varepsilon_{zz}] \qquad \gamma_{rz} = \frac{\tau_{rz}}{G} \qquad G = \frac{E}{2(1+\nu)}$$

$$\sigma_{zz} = \frac{2G}{(1-2\nu)}[(1-\nu)\varepsilon_{zz} + \nu\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}] \qquad \gamma_{z\theta} = \frac{\tau_{z\theta}}{G}$$

Axi-symmetric problems

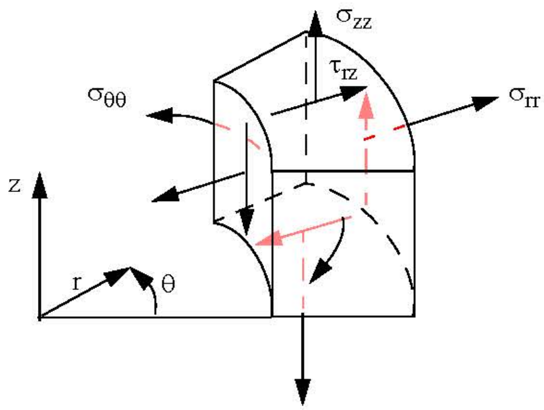
For a problem to be axi-symmetric the following requirements must be met:

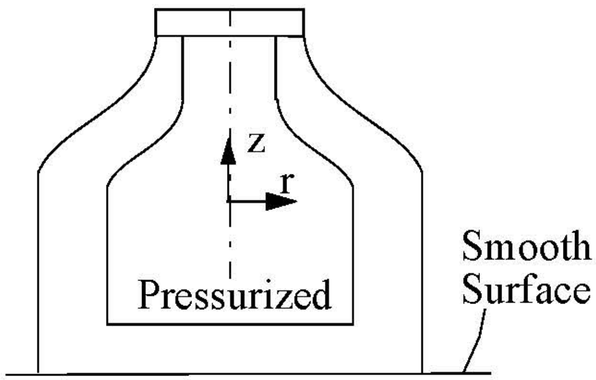
1. The geometry must be symmetric about an axis of revolution.
2. The material properties must be symmetric about the axis of revolution.
3. The loading and boundary conditions must be symmetric about the axis of revolution.

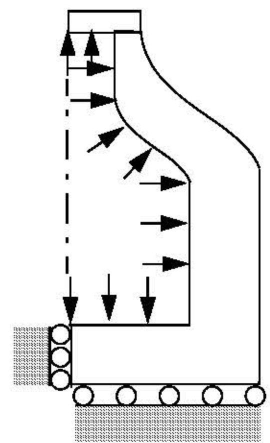
Implications: Displacements and stresses must be independent of angular location (θ) and there can be no twist (v_θ must be zero).

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad \gamma_{r\theta} = 0 \qquad \gamma_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u_r}{\partial z} \qquad \gamma_{z\theta} = 0$$

- Note radial displacement causes tangential normal strain.



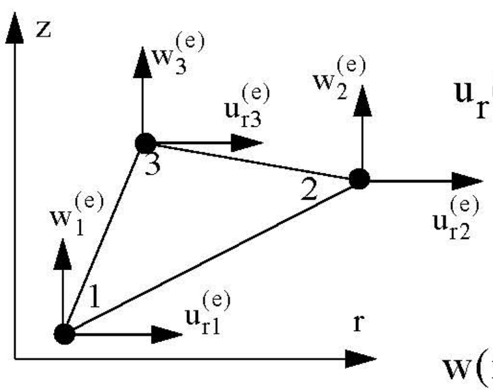




3-node Triangular Element

- Displacements are linear in r and z directions

$$u_r = a_0 + a_1 r + a_2 z \qquad w = b_0 + b_1 r + b_2 z$$



$$u_r(r, z) = \sum_{i=1}^{3} N_i(r, z) u_i^{(e)}$$

$$w(r, z) = \sum_{i=1}^{3} N_i(r, z) v_i^{(e)}$$

$$\{d^{(e)}\} = \begin{Bmatrix} u_{r1}^{(e)} \\ w_1^{(e)} \\ u_{r2}^{(e)} \\ w_2^{(e)} \\ u_{r3}^{(e)} \\ w_3^{(e)} \end{Bmatrix}$$

$\varepsilon_{rr} = \dfrac{\partial u_r}{\partial x} = a_1 \qquad \varepsilon_{zz} = \dfrac{\partial w}{\partial z} = b_2 \qquad \gamma_{rz} = \dfrac{\partial u_r}{\partial z} + \dfrac{\partial w}{\partial r} = a_2 + b_1$ —Same as CST

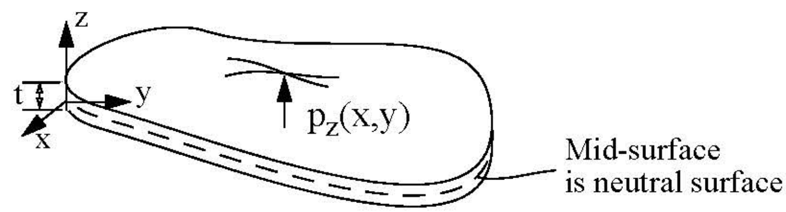
$\varepsilon_{\theta\theta} = \dfrac{u_r}{r} = \dfrac{a_0}{r} + a_1 + \dfrac{a_2 z}{r}$ ——— Tangential normal strain is not constant

- You can use any 2-D element, but will need to post-process the results of displacements and strains to get $\varepsilon_{\theta\theta}$, σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}.

Thin Plate

A thin two-dimensional structural element that is subjected to bending loads.

- Plane stress in z-direction



- Mid-plane is initially flat
- Plane sections before deformation remain plane after deformation. (displacements u and v are linear in z, i.e., through the thickness.)

Kirchhoff Plate Theory

- Plane sections initially perpendicular to the mid-surface remains perpendicular after deformation$\gamma_{xz} \approx 0$ $\gamma_{yz} \approx 0$ () --Shearing action is small)

$$u = -z\frac{\partial w}{\partial x} \qquad v = -z\frac{\partial w}{\partial y}$$

w is the displacement in the z-direction and is only a function of x and y. u and v are displacements in x and y direction.

For small strain:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z\frac{\partial^2 w}{\partial x \partial y}$$

Stresses in plane stress:

$$\sigma_{xx} = E\frac{[\varepsilon_{xx} + \nu\varepsilon_{yy}]}{(1-\nu^2)} = \frac{-Ez}{(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_{yy} = E\frac{[\varepsilon_{yy} + \nu\varepsilon_{xx}]}{(1-\nu^2)} = \frac{-Ez}{(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = \frac{-Ez}{2(1+\nu)}\frac{\partial^2 w}{\partial x \partial y}$$

Internal Forces and Moments:

$$M_x = \int_{-t/2}^{t/2} z\sigma_{xx}\, dz \qquad M_y = \int_{-t/2}^{t/2} z\sigma_{yy}\, dz \qquad M_{xy} = \int_{-t/2}^{t/2} z\tau_{xy}\, dz$$

$$q_x = \int_{-t/2}^{t/2} \tau_{xz}\, dz \qquad q_y = \int_{-t/2}^{t/2} \tau_{yz}\, dz$$

The moments and shear forces have units of moments and forces per unit length.

Moment Curvature Formulas:

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right] \qquad M_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right] \qquad M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

where, $D = \dfrac{Et^3}{12(1-\nu^2)}$ is called the plate rigidity.

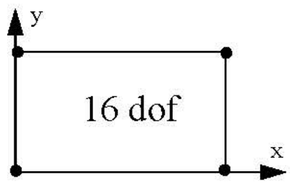
Differential Equation: Bi-harmonic Equation

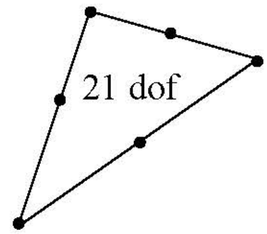
$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = p_z(x, y) \text{ or } \nabla^4 w = \nabla^2\nabla^2 w = p_z(x, y)$$

where, $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the harmonic operator.

- A kinematically admissible deflection w requires continuity of w, $\dfrac{\partial w}{\partial x}$, $\dfrac{\partial w}{\partial y}$ at all points.
- At a corner the requirement that $\dfrac{\partial^2 w}{\partial x \partial y} = \dfrac{\partial^2 w}{\partial y \partial x}$ results in the condition that $\dfrac{\partial^2 w}{\partial x \partial y}$ be continuous at the corner.

- Rectangular element: Each node has four degrees of freedom (dof) per node: w, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, $\frac{\partial^2 w}{\partial x \partial y}$. Can be used only with rectangular sides parallel to x and y axis. Hermite polynomials are used for interpolation functions.





- To ensure $\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x}$ at any orientation, requires all second derivatives to be continuous at nodes.
- Triangular element: Each corner node has six degrees of freedom per node w, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$, $\frac{\partial^2 w}{\partial x \partial y}$ and the middle node on each side has one degree of freedom $\frac{\partial w}{\partial n}$ where the n direction is the normal direction to the side.
- The continuity of second derivatives implies that moments must be continuous. If there is a line load of moment then this will leads to problems.
- Non-conforming elements do not satisfy all continuity requirements. Non-conforming elements are used in plate analysis.

Mindlin Plate Theory

- Mindlin plate theory differs from Kirchhoff plate theory in the same way as Timoshenko's beam theory differs from classical beam theory.
- The assumption of plane sections initially perpendicular to the mid-surface remains perpendicular after deformation is dropped and transverse shear is accounted.

Displacements:

$$u = z\theta_y \qquad v = -z\theta_x$$

where, θ_x and θ_y are the rotation about x and y axis, respectively, of a line that was initially perpendicular to the mid surface.

Strains

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = z\frac{\partial \theta_y}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial \theta_x}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = z\left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right)$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \left(\frac{\partial w}{\partial x} + \theta_y\right) \qquad \gamma_{yz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \left(\frac{\partial w}{\partial y} - \theta_x\right)$$

- Note $\theta_y = -\left(\frac{\partial w}{\partial x}\right)$ and $\theta_x = \frac{\partial w}{\partial y}$ reduces Mindlin's theory to Kirchhoff's theory.
- Kinematically admissibility requires that $w, \ \theta_x, \ \theta_y$ must be continuous. Can use Lagrange polynomial for interpolation functions.

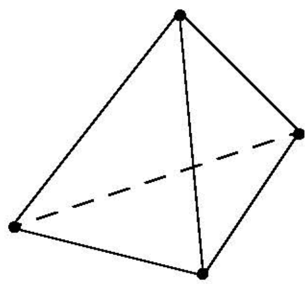
Thin Shell Elements

- Curved plate: Combination of membrane (2-D in-plane) and plate bending.
- The elements are similar to plate elements but requires definition of curved geometry.
- FEM codes usually have shallow thin shell elements which can be used to also simulate plate elements.

Three Dimensional Elements

Tetrahedron

- Displacements are linear in x and y, resulting in constant strains.

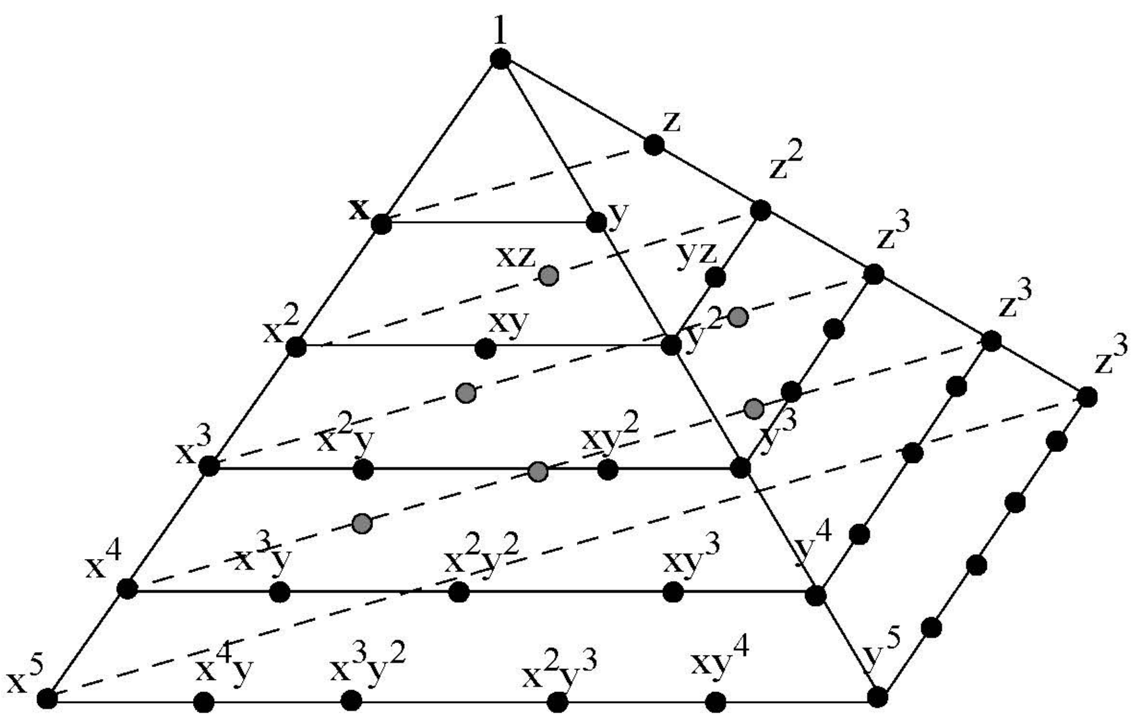


Constant Strain

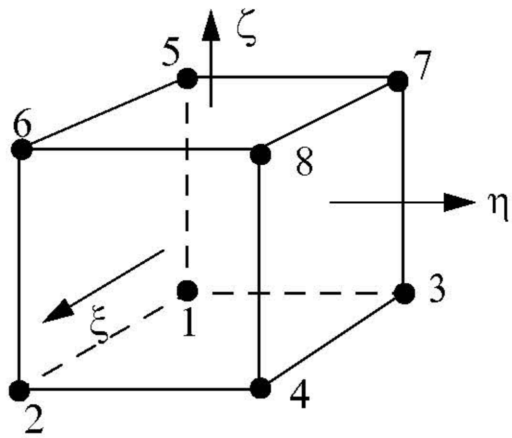
$$u = a_0 + a_1 x + a_2 y + a_3 z$$

$$v = b_0 + b_1 x + b_2 y + b_3 z$$

$$w = c_0 + c_1 x + c_2 y + c_3 z$$



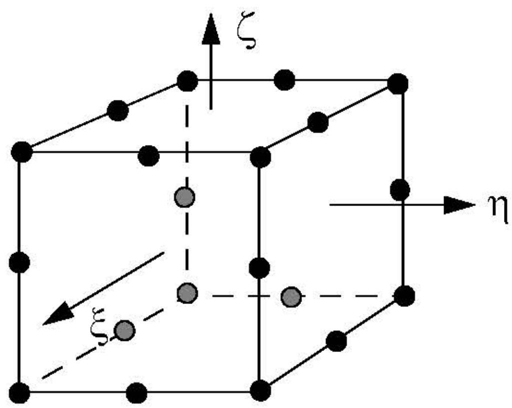
Hexahedron (Brick) Element



Tri-linear

$$N_1 = \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)$$

$$N_4 = \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)$$



20 node quadratic

Iso-parametric:

$$u = \sum_{i=1} N_i(\xi, \eta, \zeta)u_i^{(e)} \qquad x = \sum_{i=1} N_i(\xi, \eta, \zeta)x_i$$

$$v = \sum_{i=1}^{n} N_i(\xi, \eta, \zeta)v_i^{(e)} \qquad y = \sum_{i=1}^{n} N_i(\xi, \eta, \zeta)y_i$$

$$w = \sum^{n} N_i(\xi, \eta, \zeta)w_i^{(e)} \qquad z = \sum^{n} N_i(\xi, \eta, \zeta)z_i$$

$$[K^{(e)}] = \iiint [B]^T[E][B](dx)(dy)(dz) = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [\tilde{B}]^T[E][\tilde{B}]|J|(d\xi)(d\eta)(d\zeta)$$