FEM in two-dimension

Strain energy density: $U_o = \frac{1}{2}[\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \tau_{xy}\gamma_{xy}]$

Define: $\{\tilde{\sigma}\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} \qquad \{\tilde{\varepsilon}\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \quad U_o = \frac{1}{2}\{\tilde{\sigma}\}^T\{\tilde{\varepsilon}\}$

Generalized Hooke's law

Plane stress (All stresses with subscript z are zero)
Plane strain (All strains with subscript z are zero)

Plane Stress

$$\sigma_{xx} = E\frac{[\varepsilon_{xx} + \nu\varepsilon_{yy}]}{(1-\nu^2)}$$

$$\sigma_{yy} = E\frac{[\varepsilon_{yy} + \nu\varepsilon_{xx}]}{(1-\nu^2)}$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

Plane Strain

$$\sigma_{xx} = E\frac{[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}]}{(1+\nu)(1-2\nu)}$$

$$\sigma_{yy} = E\frac{[(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{xx}]}{(1+\nu)(1-2\nu)}$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

Plane Stress

$$[\tilde{E}] = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

Plane Strain

$$[\tilde{E}] = \frac{E}{(1-2\nu)(1+\nu)}\begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix}$$

$$\{\tilde{\sigma}\} = [\tilde{E}]\{\tilde{\varepsilon}\} \qquad [\tilde{E}] = [\tilde{E}]^T$$

$$E \longleftrightarrow E/(1-\nu^2)$$

$$\nu \longleftrightarrow \nu/(1-\nu)$$

$$U_o = \frac{1}{2}\{\tilde{\sigma}\}^T\{\tilde{\varepsilon}\} = \frac{1}{2}\{\tilde{\varepsilon}\}^T[\tilde{E}]^T\{\tilde{\varepsilon}\} = \frac{1}{2}\{\tilde{\varepsilon}\}^T[\tilde{E}]\{\tilde{\varepsilon}\}$$

Strain-Displacement

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\{\varepsilon\} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

Displacements approximating:

$$u(x) = \sum_{i=1}^{n} u_i^{(e)} f_i(x, y) \qquad v(x) = \sum_{i=1}^{n} v_i^{(e)} f_i(x, y)$$

Define the nodal displacement vector as: $\{d\} = \begin{Bmatrix} u_1^{(e)} \\ v_1^{(e)} \\ u_2^{(e)} \\ v_2^{(e)} \\ \bullet \\ \bullet \\ u_n^{(e)} \\ v_n^{(e)} \end{Bmatrix}$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} f_1 & 0 & f_2 & 0 & \dots & \dots & f_n & 0 \\ 0 & f_1 & 0 & f_2 & \dots & \dots & 0 & f_n \end{bmatrix} \{d\}$$

$$\{\varepsilon\} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} f_1 & 0 & f_2 & 0 & \dots & \dots & f_n & 0 \\ 0 & f_1 & 0 & f_2 & \dots & \dots & 0 & f_n \end{bmatrix} \{d\} = [B]\{d\}$$

$$[B] = \begin{bmatrix} \frac{\partial f_1}{\partial x} & 0 & \frac{\partial f_2}{\partial x} & 0 & \dots & \dots & \frac{\partial f_n}{\partial x} & 0 \\ 0 & \frac{\partial f_1}{\partial y} & 0 & \frac{\partial f_2}{\partial y} & \dots & \dots & 0 & \frac{\partial f_n}{\partial y} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial x} & & & \frac{\partial f_n}{\partial y} & \frac{\partial f_n}{\partial x} \end{bmatrix}$$

- Matrix [B] is called strain-displacement matrix

$$U_o^{(e)} = \frac{1}{2}\{d\}^T\{B\}^T[\tilde{E}]\{B\}\{d\}$$

Strain Energy: $U^{(e)} = \int_V U_o^{(e)} dV$

$$U^{(e)} = \int_V \frac{1}{2}\{d\}^T\{B\}^T[\tilde{E}]\{B\}\{d\}dV = \frac{1}{2}\{d\}^T[K^{(e)}]\{d\}$$

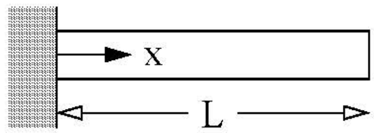
Element stiffness matrix: $[K^{(e)}] = \int_V [B]^T[E][B]\ dV$

Variation in potential energy: $\delta\Omega^{(e)} = \{\delta d\}^T([K^{(e)}]\{d\} - \{R^e\})$

Overview of approximate methods

Some Jargon

1-D Heat Conduction:

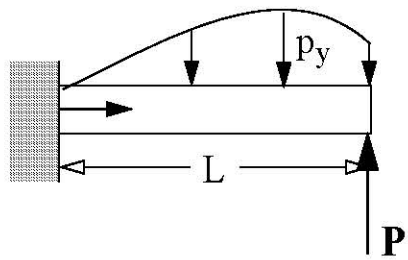


$$\frac{d^2T}{dx^2} = 0 \qquad 0 \le x \le L$$ —— *Differential Equation*

$$-k\frac{dT}{dx}\bigg|_{x=0} = 0$$ —— *Natural Boundary Condition*

$$T\big|_{x=L} = T_o$$ —— *Essential Boundary Condition*

Beam Bending



$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = p_y \qquad 0 \le x \le L$$ — *Differential Equation*

$$V = -\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = P$$

$$M = EI\frac{d^2v}{dx^2} = 0$$

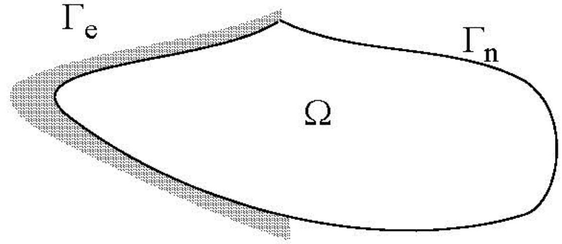
— *Natural Boundary Condition*

$$v(0) = 0$$

$$\left.\frac{dv}{dx}\right|_{x=0} = 0$$

— *Essential Boundary Condition*

Approximation of boundary value problem



$Lu = f$ in Ω ——— *Differential Equation*

$D_n u = g_n$ on Γ_e ——— *Natural Boundary Condition*

$D_e u = g_e$ on Γ_n ——— *Essential Boundary Condition*

$$u = \sum_{i=1}^{n} c_j \phi_j$$

ϕ_j set of approximating functions

set of ϕ_j is complete and independent.

$$e_d = \sum_{j=1}^{n} c_j L\phi_j - f$$ ——*Error in Differential Equation*

$$e_n = \sum_{j=1}^{n} c_j D_n\phi_j - g_n$$ ——*Error in Natural Boundary Condition*

$$e_e = \sum_{j=1}^{n} c_j D_e\phi_j - g_e$$ ——*Error in Essential Boundary Condition*

Commonality and Differences in Approximate Methods

Commonalities

- Produce a set of algebraic equations in the unknown constants c_j.
- Choose ϕ_i to set one (or two) of the errors e_d, e_e, or e_n to zero
- Minimize the remaining error(s).

Differences

- Which error is set to zero

  Domain Methods: $e_e = 0$ or $e_n = 0$

  Boundary Methods: $e_d = 0$
- Error Minimizing Process

Independence of ϕ_i

- No ϕ_i can be obtained from a linear combination of other ϕ_i's in the set.
- If the set of functions ϕ_i are not independent then the equations in the matrix will not be independent and the matrix will be singular.

Completeness of ϕ_i

- In a series sequence no term should be skipped.
- If a set is not complete then the solution may not converge for some problems.

Error Minimization

Weighted Residue

$$\iint_{\Omega} \psi_i^{(d)} e_d \, dx\,dy + \int_{\Gamma_e} \psi_i^{(e)} e_e \, ds + \int_{\Gamma_n} \psi_i^{(n)} e_n \, ds = 0$$

FEM-Stiffness version: $e_e = 0$

$$\iint_{\Omega} \psi_i^{(d)} e_d \, dx\,dy + \int_{\Gamma_n} \psi_i^{(n)} e_n \, ds = 0$$

FEM-Flexibility version: $e_n = 0$

$$\iint_{\Omega} \psi_i^{(d)} e_d \, dx\,dy + \int_{\Gamma_e} \psi_i^{(e)} e_e \, ds = 0$$

BEM: $e_d = 0$

$$\int_{\Gamma_n} \psi_i^{(n)} e_n \, ds + \int_{\Gamma_e} \psi_i^{(e)} e_e \, ds = 0$$

FEM: Discretization process is on domain of the entire body Ω

BEM: Discretization process is on the boundary of the body Γ

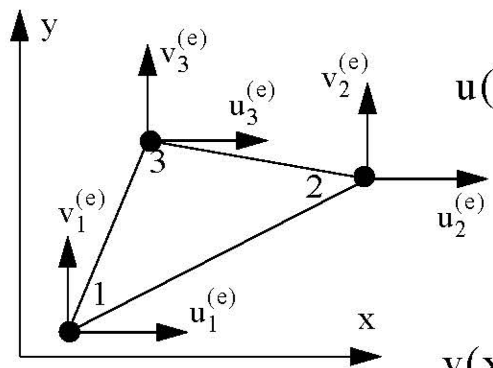
- In FEM stiffness matrix the equilibrium equation on stresses (differential equations) and boundary conditions on stresses (natural boundary conditions) are approximately satisfied.
- In FEM flexibility matrix the compatibility equation on stresses (differential equations) and boundary conditions on displacements (essential boundary conditions) are approximately satisfied.

Constant Strain Triangle (CST)

- Displacements are linear in x and y, resulting in constant strains.

$$u = a_0 + a_1 x + a_2 y \qquad v = b_0 + b_1 x + b_2 y$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = a_1 \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = b_2 \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_2 + b_1$$



$$u(x, y) = \sum_{i=1}^{3} N_i(x, y) u_i^{(e)}$$

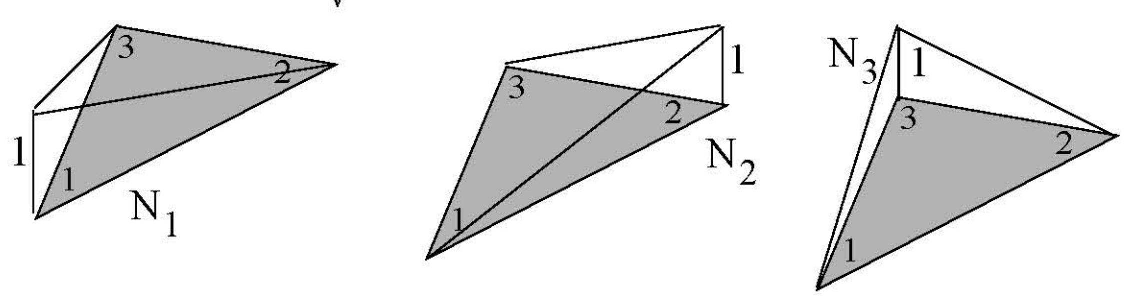
$$v(x, y) = \sum_{i=1}^{3} N_i(x, y) v_i^{(e)}$$

$$\{d^{(e)}\} = \begin{Bmatrix} u_1^{(e)} \\ v_1^{(e)} \\ u_2^{(e)} \\ v_2^{(e)} \\ u_3^{(e)} \\ v_3^{(e)} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & & x_{13} & & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}\{d^{(e)}\} = [B]\{d^{(e)}\}$$

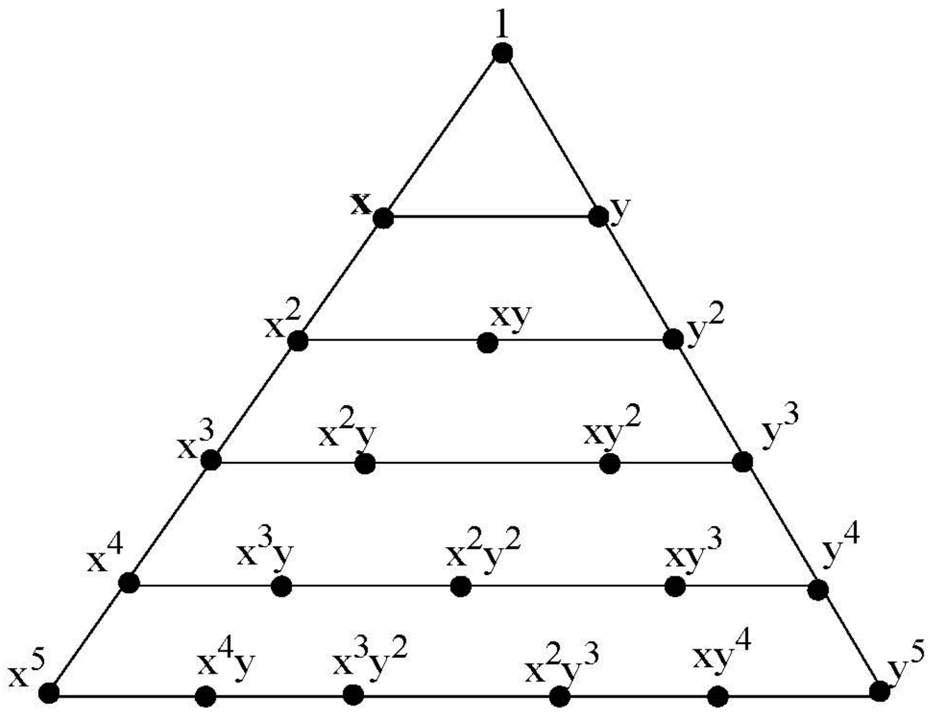
$$x_{ij} = x_i - x_j \qquad y_{ij} = y_i - y_j$$

$$[K^{(e)}] = \int_V [B]^T[E][B]\ dV = [B]^T[E][B]tA$$



Pascal's Triangle

- Used for determining a complete set of polynomial terms in two dimensions.'

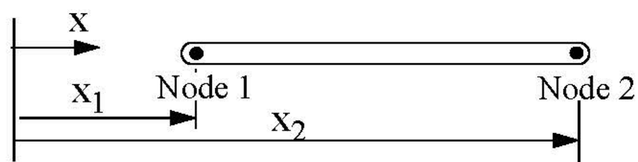


- Greater the degree of freedom, less stiff will be element.
- Interpolation functions are easier to develop with area coordinates.

Natural Coordinates

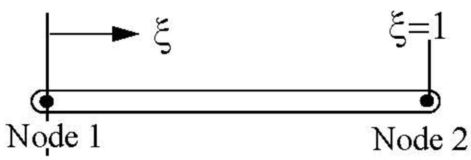
- Coordinates which vary between 0 and 1 or -1 and 1.
- Natural coordinates and non-dimensional coordinates.

1-d Coordinates



$$L_1(x) = \left(\frac{x - x_2}{x_1 - x_2}\right) \qquad L_2(x) = \left(\frac{x - x_1}{x_2 - x_1}\right)$$

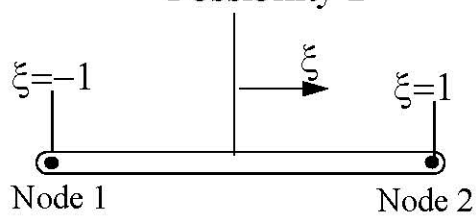
Possibility 1



$$L_1(\xi) = (1 - \xi)$$

$$L_2(\xi) = \xi$$

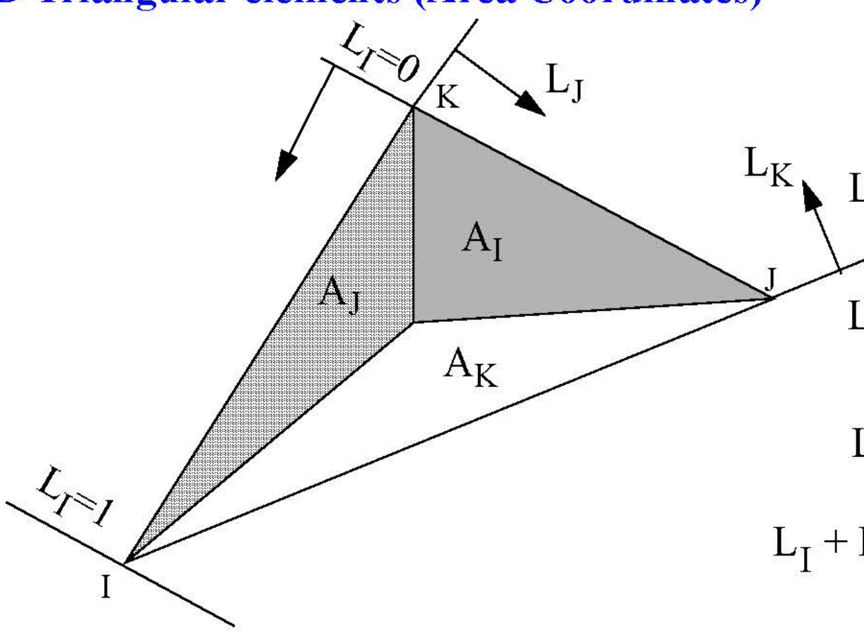
Possibility 2



$$L_1(\xi) = (1 - \xi)/2$$

$$L_2(\xi) = (1 + \xi)/2$$

2-D Triangular elements (Area Coordinates)



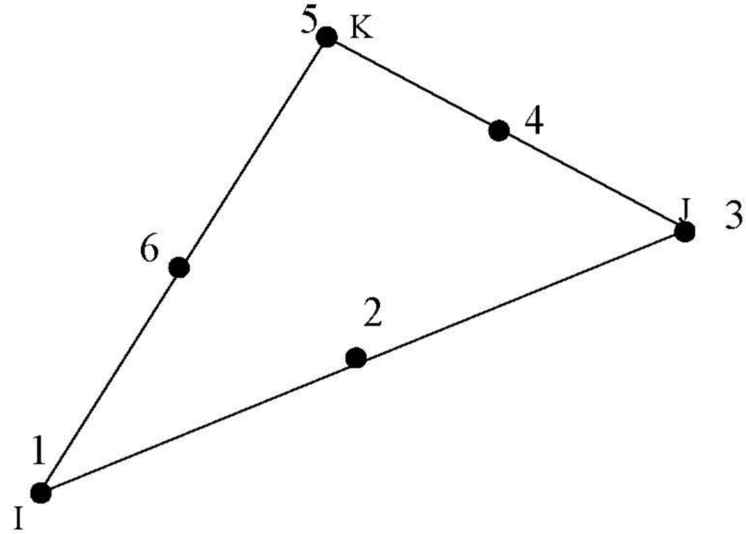
$$L_I = \frac{A_I}{A}$$

$$L_J = \frac{A_J}{A}$$

$$L_K = \frac{A_K}{A}$$

$$L_I + L_J + L_K = 1$$

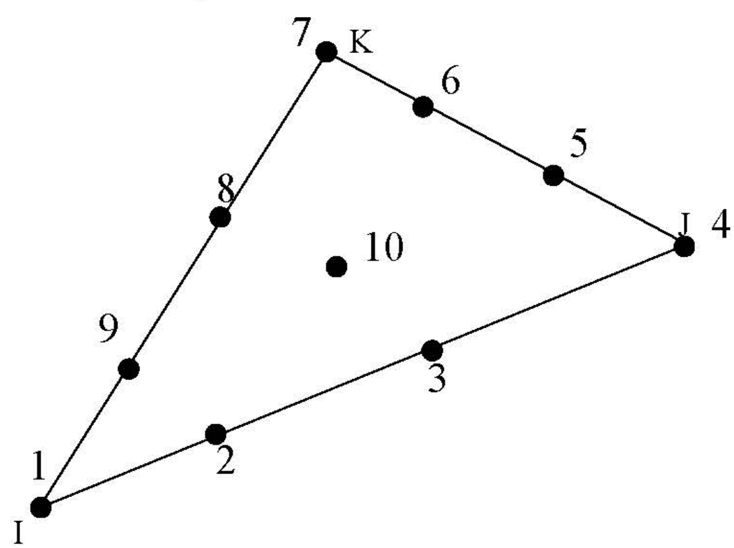
Linear Strain Triangle



$$N_1 = \frac{L_I(L_I - 1/2)}{(1)(1/2)} = 2L_I\left(L_I - \frac{1}{2}\right)$$

$$N_2 = \frac{L_I L_J}{(1/2)(1/2)} = 4L_I L_J$$

Homework Problem: Write cubic interpolation functions using area coordinates for nodes 1,2 and 10.

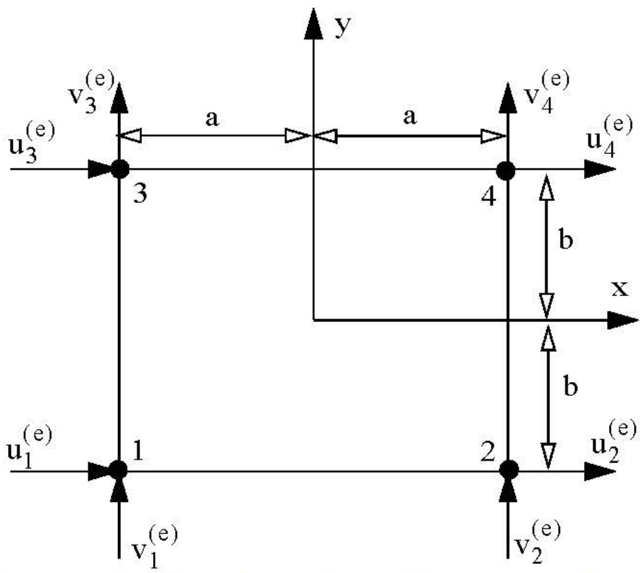


Bi-Linear Quadrilateral

$$u = a_0 + a_1 x + a_2 y + a_3 xy \qquad v = b_0 + b_1 x + b_2 y + b_3 xy$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = a_1 + a_3 y \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = b_2 + b_3 x$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = (a_2 + b_1) + a_3 x + b_3 y$$



$$u(x, y) = \sum_{i=1}^{4} N_i(x, y) u_i^{(e)}$$

$$v(x, y) = \sum_{i=1}^{4} N_i(x, y) v_i^{(e)}$$

Interpolation functions in natural coordinates

$$\xi = x/a \qquad \eta = y/b$$

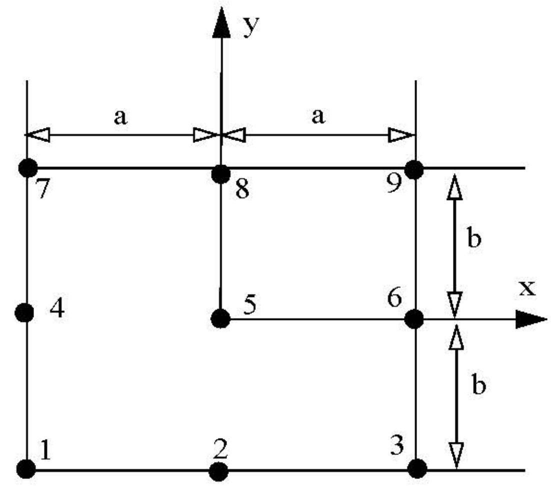
$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1-\xi)(1+\eta) \qquad N_4 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{a}\frac{\partial N_i}{\partial \xi} \qquad \frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{b}\frac{\partial N_i}{\partial \eta}$$

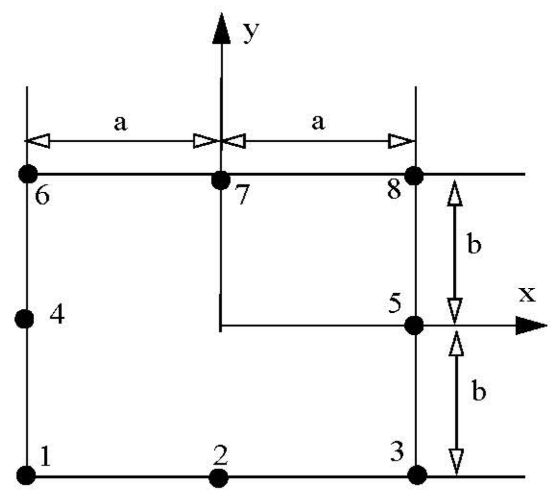
Other Quadrilaterals

Complete quadratic



- The stiffness (row and column) related to node 5 is known at element level and as rows and columns of other elements do not add to it.

Quadratic element often used in practice:



- When internal nodes are eliminated care has to be exercised to ensure the mesh from such elements will converge.

Mechanical Loads

There are three types of mechanical loads

1. Concentrated Forces or Moments

- The loads must be applied at nodes when making the mesh.
- Theoretically the stresses are infinite at the point of application, hence in the neighborhood of concentrated load a large stress gradient can be anticipated.

2. Tractions

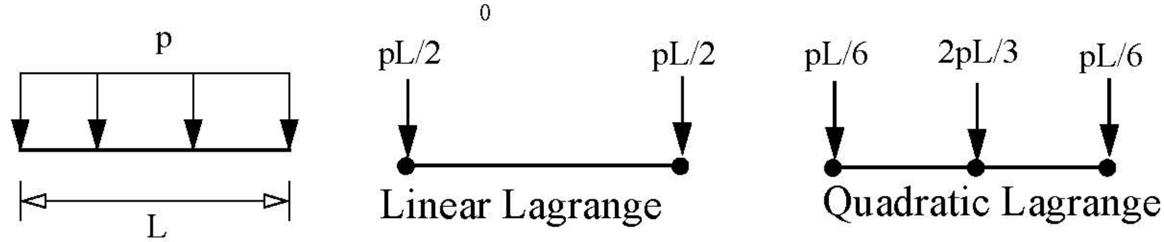
- Forces that act on the bounding surfaces.

$$S_x = \sigma_{xx}n_x + \tau_{xy}n_y \qquad S_y = \tau_{yx}n_x + \sigma_{yy}n_y$$

n_x and n_y are the direction cosines of the unit normal

- Tractions has units of force per unit area and are distributed forces.
- Usually the tractions are specified in local normal and tangential coordinates.
- These distributed forces must be converted to nodal forces.

In two dimensions the bounding surface is a curve. Distributed forces can be converted to nodal forces as was done in 1-d axial and bending problems. (work equivalency) $R_j = \int_0^L p(x)f_j(x)\ dx$



3. Body Forces

- Forces that act at each and every point on the body.
- Gravity, magnetic, inertial are some examples.
- These forces must be converted to nodal forces.

(See section 3.9)