Variational Calculus in 1-D

The learning objectives in this chapter are:

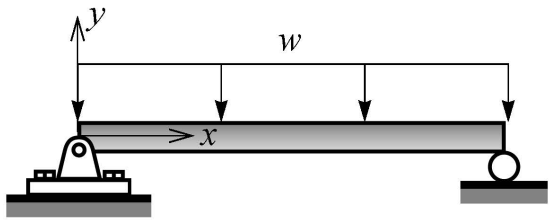
- Understand the concepts in variational calculus.
- Understand the application of variational calculus to obtain boundary value problems in mechanics of materials.

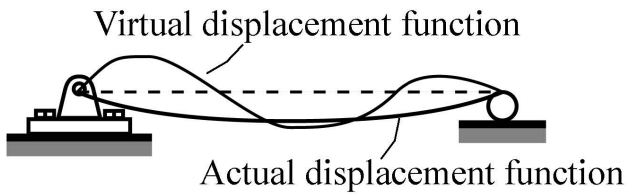
Basic Concepts

Variational calculus is the branch of mathematics dealing with finding the maximum and minimum values of functionals.

- A **function** $u(x)$ is a rule of correspondence such that for all x in D there is assigned a unique element $u(x)$ in R. A **functional** $F[u(x)]$ is a rule of correspondence such that for all $u(x)$ in R there is assigned a unique element $F[u(x)]$ in Ω In other words, a functional is a function of a function. Strain energy, the potential of work, and potential energy are all functions of the displacements which are functions of the position coordinates.

1. We can walk to different points and measure the elevation— we will call it the d-process; so dx, ds, du represent the **actual movement** along a path (curve).
2. We can conduct a thought experiment. For example, without moving, we ask the question, if we go to that point will the elevation increase or decrease? This imaginary movement is called the **virtual movement** or the δ-process.





Independent set of functions

$$a_1u_1 + a_2u_2 + a_3u_3 \bullet \bullet \bullet \bullet + a_nu_n = 0$$

If $u_1, u_2, u_3, \ldots u_n$ are independent functions then above equation implies $a_i = 0$ $i = 1$ to n

- Any set of independent variables (parameters) that describes the system geometry are called the **generalized coordinates.**
- The space spanned by the generalized coordinates is called the **configuration space**.
- Any condition that limits the change in geometry in the configuration space is called the **kinematic condition.**
- Functions that are *continuous* and satisfy *all* the *kinematic boundary conditions* are called **kinematically admissible functions**.

Extremum and Stationary Values

Find minimum of $F = F(u_1, u_2, \bullet\ \bullet\ \bullet + u_n)$

We consider a virtual change in the configuration space, that is, space spanned by the independent variables u_1, u_2, u_3,...u_n. The total virtual change δF is the sum of the slopes multiplied by virtual change in each direction.

$$\delta F = \frac{\partial F}{\partial u_1}\delta u_1 + \frac{\partial F}{\partial u_2}\delta u_2 + \frac{\partial F}{\partial u_3}\delta u_3 \bullet\ \bullet\ \bullet\ \bullet + + \frac{\partial F}{\partial u_n}\delta u_n$$

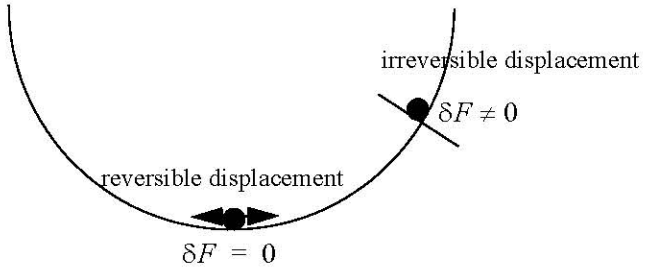
- δF is called the **first variation** of F. If F is to be a minimum at a point in the configuration space, then this change of δF must be zero.

$$\delta F = \frac{\partial F}{\partial u_1}\delta u_1 + \frac{\partial F}{\partial u_2}\delta u_2 + \frac{\partial F}{\partial u_3}\delta u_3 \bullet\ \bullet\ \bullet\ \bullet + + \frac{\partial F}{\partial u_n}\delta u_n = 0$$

If u_1, u_2, u_3,...u_n are independent variables then we are free to move in any direction. So if we only walk in u_1 (all other virtual displacement are zero) then we have $\partial F/\partial u_1 = 0$. In a similar manner we can walk in each of the directions and conclude:

$$\frac{\partial F}{\partial u_i} = 0 \qquad i = 1, 2, 3 \bullet\ \bullet\ \bullet n$$

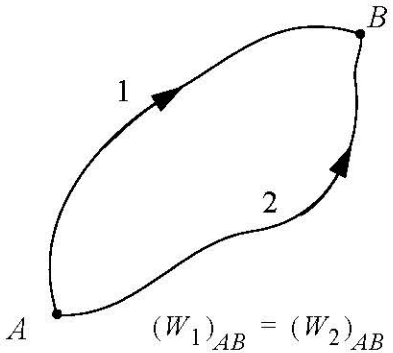
- For stationary values we need the virtual displacement to be **reversible**.



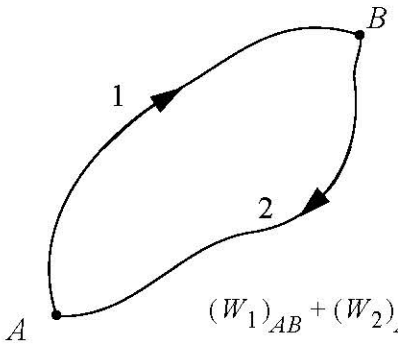
Work

Work is done by a force if the point at which the force is applied moves. If the point at which force $\overline{F}$ is applied moves through an infinitesimal distance $d\overline{u}$, then the work is defined as

$$dW = \overline{F} \cdot d\overline{u}$$



$(W_1)_{AB} = (W_2)_{AB}$



$(W_1)_{AB} + (W_2)_{AB} = 0$

- Work done by a force is **conserved** if it is path independent.
- Friction, permanent deformation are examples in which work will not be conserved.
- Rubber has nonlinear stress–strain curve. Work done in stretching rubber is recovered when the forces are released and the rubber returns to the undeformed position.

Nonlinear systems and non-conservative systems are two independent descriptions of a system.

- Work is a scaler quantity. Work from different types of forces and moments can be added.

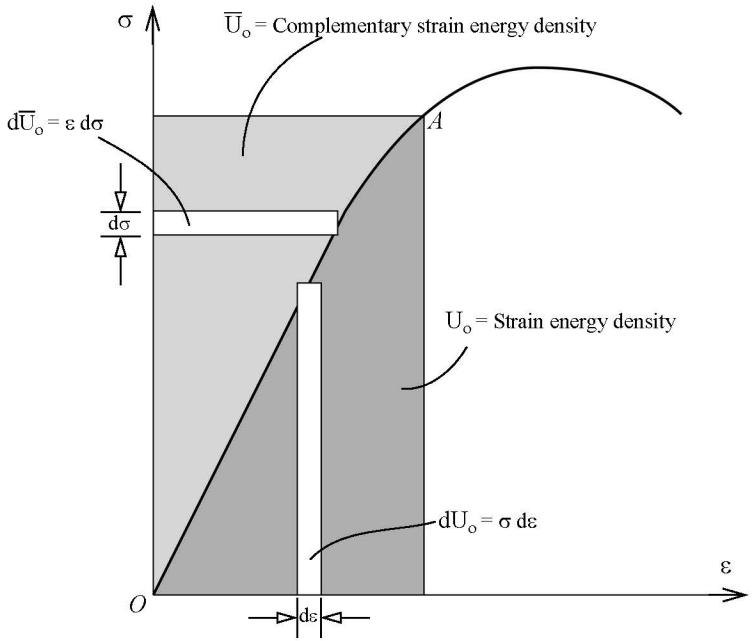
Table 1: Work Expressions

	Work
Axial	$W_A = \int_L p_x(x)u(x)dx + \sum_{q=1}^{m} F_q u(x_q) = l(u)$
Torsion of circular shafts	$W_T = \int_L t(x)\phi(x)dx + \sum_{q=1}^{m} T_q \phi(x_q) = l(\phi)$
Symmetric bending of beams	$W_B = \int_L p_y(x)v(x)dx + \sum_{q=1}^{m_1} F_q v(x_q) + \sum_{q=1}^{m_2} M_q \frac{dv}{dx}(x_q) = l(v)$

Strain Energy

- The change in internal energy in a body during deformation is called the **strain energy**.
- The energy per unit volume is called the **strain energy density** and is the area under the stress–strain curve up to the point of deformation.

$$U = \int_V U_0 \, dV \qquad U_0 = \int_0^{\varepsilon} \sigma \, d\varepsilon \qquad \overline{U}_0 = \int_0^{\sigma} \varepsilon \, d\sigma$$



- The units for strain energy density are newton-meters per cubic meter ($N \cdot m/m^3$), joules per cubic meter (J/m^3), inch-pounds per cubic inch ($in \cdot lb/in^3$), and foot-pounds per cubic foot ($ft \cdot lb/ft^3$).

Linear Strain Energy Density

$$U_0 = \frac{1}{2}\sigma\varepsilon \qquad U_0 = \frac{1}{2}\tau\gamma$$

- Strain energy, hence strain energy density, is a scalar quantity. We can add the strain energy density due to individual stress and strain components to obtain the total linear strain energy density during deformation.

$$\boxed{U_0 = \frac{1}{2}[\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}]}$$

Linear strain energy in symmetric bending of beams

Two nonzero stress components, σ_{xx} and τ_{xy}. $\sigma_{xx} = E\varepsilon_{xx}$ and $\varepsilon_{xx} = -y(d^2\mathrm{v}/dx^2)$

$$U_B = \int_V \frac{1}{2}E\varepsilon_{xx}^2\,dV = \int_L \left[\int_A \frac{1}{2}E\left(y\frac{d^2\mathrm{v}}{dx^2}\right)^2 dA\right]dx = \int_L \left[\frac{1}{2}\left(\frac{d^2\mathrm{v}}{dx^2}\right)^2 \int_A Ey^2\,dA\right]dx = \frac{1}{2}\int_L EI_{zz}\left(\frac{d^2\mathrm{v}}{dx^2}\right)^2 dx$$

The strain energy due to shear in bending is $U_S = (1/2)\int \tau_{xy}\gamma_{xy}dV$.

The maximum shear stress τ_{xy} and shear strain γ_{xy} are an order of magnitude smaller than the maximum normal stress σ_{xx} and the maximum normal strain ε_{xx}. U_S will be two orders of magnitude smaller than U_B and can be neglected in our calculations.

Table 2: Strain Energy and bilinear functional form of strain energy

	Strain Energy
Axial	$U_A = \frac{1}{2}\int_L EA\left(\frac{du}{dx}\right)^2 dx; \quad U_A = \frac{1}{2}\int_L \left[EA\frac{du_1}{dx}\frac{du_2}{dx}\right]dx = \frac{1}{2}B(u_1, u_2)$
Torsion of circular shafts	$U_T = \frac{1}{2}\int_L GJ\left(\frac{d\phi}{dx}\right)^2 dx; \quad U_T = \frac{1}{2}\int_L \left[GJ\frac{d\phi_1}{dx}\frac{d\phi_2}{dx}\right]dx = \frac{1}{2}B(\phi_1, \phi_2)$
Symmetric bending of beams	$U_B = \frac{1}{2}\int_L EI_{zz}\left(\frac{d^2\mathrm{v}}{dx^2}\right)^2 dx; \quad U_B = \frac{1}{2}\int_L \left[EI_{zz}\frac{d^2\mathrm{v}_1}{dx^2}\frac{d^2\mathrm{v}_2}{dx^2}\right]dx = \frac{1}{2}B(\mathrm{v}_1, \mathrm{v}_2)$

Table 2: Strain Energy and bilinear functional form of strain energy

	Strain Energy
Thin Plates	$U_P = \frac{D}{2}\iint_A \left\{ \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right\} dx\,dy$
	$U_P = \frac{D}{2}\iint_A \left[\frac{\partial^2 w_1}{\partial x^2}\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}\frac{\partial^2 w_2}{\partial y^2} + \nu\left(\frac{\partial^2 w_1}{\partial x^2}\frac{\partial^2 w_2}{\partial y^2} + \frac{\partial^2 w_2}{\partial x^2}\frac{\partial^2 w_1}{\partial y^2} \right) + 2(1-\nu)\frac{\partial^2 w_1}{\partial x \partial y}\frac{\partial^2 w_2}{\partial x \partial y} \right] dx\,dy$
	$= \frac{1}{2}B(w_1, w_2)$
Plane Stress Elasticity	$U_E = \frac{Eh}{2(1-\nu^2)}\iint_A \left[\left\{ \left(\frac{\partial u}{\partial x}\right)^2 + 2\nu\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial v}{\partial y}\right)^2 \right\} + \frac{(1-\nu)}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right] dx\,dy$
	$U_E = \frac{Eh}{2(1-\nu^2)}\iint_A \left[\left\{ \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x} + \nu\left(\frac{\partial u_2}{\partial x}\frac{\partial v_1}{\partial y} + \frac{\partial u_1}{\partial x}\frac{\partial v_2}{\partial y} \right) + \frac{\partial v_1}{\partial y}\frac{\partial v_2}{\partial y} \right\} + \frac{(1-\nu)}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right)\left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x}\right) \right] dx\,dy$
	$= \frac{1}{2}B(u_1, v_1, u_2, v_2)$

Virtual Work

- Virtual work methods are applicable to linear and nonlinear systems, to conservative as well as non-conservative systems.

The total virtual work done on a body at equilibrium is zero.

- Virtual work implies it is not actual work but work done by actual forces in moving points through *virtual displacements*, or, virtual forces moving through actual displacement.

$$\delta W = 0 \qquad \delta W_{\text{ext}} = \delta W_{\text{int}}$$

Minimum Potential Energy

- We define the potential energy function Ω as

$$\Omega = U - W \tag{6.1}$$

U is the strain energy and W is the work potential of a force.

- The "work potential of a force" is associated with conservative forces only and implies that there is a potential function from which such a force can be obtained.
- Minimum potential energy and methods derived from it are applicable to conservative systems that can be linear or non-linear.

The internal virtual work is the variation in elastic strain energy during deformation: $\delta W_{\text{int}} = \delta U$

The external virtual work is the variation in the work potential of the force: $\delta W_{\text{ext}} = \delta W$

$$\delta W_{int} - \delta W_{ext} = \delta U - \delta W = \quad \text{or} \quad \boxed{\delta\Omega = 0}$$

The virtual variation in the potential energy function is zero—which occurs where the slopes of the potential energy function with respect to the parameters defining the potential function are zero.

Of all the kinematically admissible displacement functions, the actual displacement function is the one that minimizes the potential energy function at stable equilibrium.

There are many kinematically admissible displacement functions, and there is no requirement that these functions satisfy the equilibrium equations or the boundary conditions on forces and moments.

- The actual displacement is kinematically admissible and satisfies all the equilibrium conditions and the static boundary conditions.
- If we choose an arbitrary kinematically admissible function and calculate the potential energy function, the value so obtained will always be greater than the value of the potential energy function at equilibrium.
- The better approximation of displacement function is the one that yields the lower potential energy.
- The greater the degrees of freedom, the lower will be the potential energy for a given set of kinematically admissible functions.

Stationary Value Of A Definite Line Integral

$$I(u) = \int_a^b H(u', u, x)\ dx \qquad \text{where} \qquad u' = du/dx$$

$$\text{First variation: } \delta I(u) = \delta \int_a^b H(u', u, x)\ dx = \int_a^b \delta H(u', u, x)\ dx = \int_a^b \left[\frac{\partial H}{\partial u'}\delta u' + \frac{\partial H}{\partial u}\delta u\right]\ dx$$

- Function and derivative are independent in virtual displacement.
- Once we have considered virtual displacement, we are now on a specific curve and function and its derivative are related, that is, no longer independent. If we are to draw any conclusion by setting $\delta I(u) = 0$, then we need to obtain an expression only in u—we perform integration by parts.

$$\delta I(u) = \int_a^b \left[\frac{\partial H}{\partial u'}\frac{d(\delta u)}{dx} + \frac{\partial H}{\partial u}\delta u\right]\ dx = 0 = \int_a^b \left[\frac{\partial H}{\partial u} - \frac{d}{dx}\left(\frac{\partial H}{\partial u'}\right)\right]\delta u\ dx + \frac{\partial H}{\partial u'}\delta u\Big|_a^b = 0$$

Possibility 1: We meet the condition $\delta I(u) = 0$ in the average or overall sense. This lead to approximate methods as the condition is not satisfied at each and every point between a and b.

Possibility 2: We require that $\delta I(u) = 0$ at each and every point between a and b. This results in boundary value problem.

- During the process of variation, the function and its derivative are independent.
- After the process of variation, the function and its derivative are no longer independent.
- Integration by parts will generate an expression only in terms of variation of the function.

Boundary Value Problem

$$\text{Differential Equation: } \boxed{\frac{\partial H}{\partial u} - \frac{d}{dx}\left(\frac{\partial H}{\partial u'}\right) = 0 \qquad a < x < b}$$

$$\text{Boundary Conditions: } \boxed{[\partial H/\partial u' = 0 \quad \text{or} \quad \delta u = 0] \quad \text{at } x = a \text{ and at } x = b}$$

The above equations are called the *Euler-Lagrange* equations.

C1.1 The potential energy for symmetric bending of beams subjected to only distributed force p_y and no concentrated force or moment is given below. Obtain the boundary value problem for deflection v(x) by minimizing the potential energy.

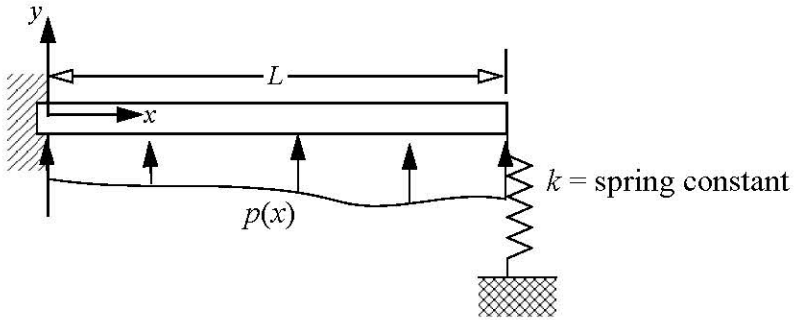
$$\Omega = \frac{1}{2}\int_L EI_{zz}\left(\frac{d^2\mathrm{v}}{dx^2}\right)^2 dx - \int_L p_y(x)\mathrm{v}(x)dx$$

where, L is the length of the beam; E is modulus of elasticity; and I_{zz} is the second area moment of inertia.

Class Problem

The presence of a linear spring changes the potential energy to the following

$$\Omega = \int_0^L \left[\frac{1}{2} EI_{zz} \left(\frac{d^2 \mathrm{v}}{dx^2} \right)^2 - p(x)\mathrm{v}(\mathrm{x}) \right] dx + \frac{1}{2} k[\mathrm{v}(L)]^2$$



Obtain the BVP for the beam shown above using variational methods.

Generalization

$$I(u) = \int_a^b H(u^{(r)}, u^{(r-1)}, u^{(r-1)}, \ \cdot\ \cdot\ \cdot\ , u^{(1)}, u^{(0)}, x)\ dx \qquad \text{where} \qquad u^{(r)} = d^r u / dx^r \qquad u^{(0)} = u$$

Differential equation:
$$\frac{\partial H}{\partial u^{(0)}} - \frac{d}{dx}\left(\frac{\partial H}{\partial u^{(1)}}\right) + \frac{d^2}{dx^2}\left(\frac{\partial H}{\partial u^{(2)}}\right) - \frac{d^3}{dx^3}\left(\frac{\partial H}{\partial u^{(3)}}\right) + \ \cdot\ \cdot\ \cdot\ + (-1)^r \frac{d^r}{dx^r}\left(\frac{\partial H}{\partial u^{(r)}}\right) = 0$$

$$-\left(\frac{\partial H}{\partial u^{(1)}}\right) + \frac{d}{dx}\left(\frac{\partial H}{\partial u^{(2)}}\right) - \frac{d^{(2)}}{dx^2}\left(\frac{\partial H}{\partial u^{(3)}}\right) + \ \bullet\ \bullet\ + (-1)^r \frac{d^{(r-1)}}{dx^{(r-1)}}\left(\frac{\partial H}{\partial u^{(r)}}\right) = 0 \qquad \text{or} \qquad \delta u = 0$$

$$\left(\frac{\partial H}{\partial u^{(2)}}\right) - \frac{d}{dx}\left(\frac{\partial H}{\partial u^{(3)}}\right) + \ \bullet\ \bullet\ + (-1)^r \frac{d^{(r-2)}}{dx^{(r-2)}}\left(\frac{\partial H}{\partial u^{(r)}}\right) = 0 \qquad \text{or} \qquad \delta u^{(1)} = 0$$

Boundary conditions at $x = a$ and $x = b$

$$\left(\frac{\partial H}{\partial u^{(3)}}\right) + \ \bullet\ \bullet\ + (-1)^r \frac{d^{(r-3)}}{dx^{(r-3)}}\left(\frac{\partial H}{\partial u^{(r)}}\right) = 0 \qquad \text{or} \qquad \delta u^{(2)} = 0$$

$$\bullet = 0 \qquad \text{or} \qquad = 0$$

$$\bullet = 0 \qquad \text{or} \qquad = 0$$

$$\bullet = 0 \qquad \text{or} \qquad = 0$$

$$\frac{\partial H}{\partial u^{(r)}} = 0 \qquad \text{or} \qquad \delta u^{(r-1)} = 0$$

1. If the highest derivative in the functional is r, then the differential equation will be of order $2r$.
2. The boundary conditions with the variation symbol of δ have derivatives from 0 to r-1. These quantities must be continuous and are called **primary variables**. The boundary conditions are called **kinematic boundary conditions** or **essential boundary conditions**.
3. The boundary conditions that have derivatives of the functionals that vary from r to $2r$-1 are our internal forces and moments and are called **statical** variables or **secondary variables.** The boundary conditions on these variables are called **statical boundary conditions** or **natural boundary conditions**.
4. If the functional contains more than one variable (u), say u_i, then we could replace u with u_i in the above equations.

- If the functional is quadratic in u and its derivative, then the boundary value problem will be linear. However, the above equations are applicable to any functional.

Rayleigh-Ritz Method (See Reddy Section 2.5)

- **Linear functional**: $l(\alpha_1 u + \alpha_2 \mathrm{v}) = \alpha_1 l(u) + \alpha_2 l(\mathrm{v})$
- **Bilinear functional** $B(\alpha_1 u_1 + \alpha_2 u_2, \mathrm{v}) = \alpha_1 B(u_1, \mathrm{v}) + \alpha_2 B(u_2, \mathrm{v}) \qquad B(u, \alpha_1 v_1 + \alpha_2 \mathrm{v}_2) = \alpha_1 B(u, \mathrm{v}_1) + \alpha_2 B(u, \mathrm{v}_2)$
- **Symmetric bilinear functional**: $B(u, \mathrm{v}) = B(\mathrm{v}, u)$; u and v can be vectors.
- Rayleigh-Ritz method is a formal process of minimizing the functional (potential energy) given below using a series of kinematically admissible displacement functions to produce a set of algebraic equations in the unknown constants of the series approximation.

$$\boxed{\Omega = \frac{1}{2} B(u, u) - l(u)}$$

- Rayleigh-Ritz method is applicable to conservative systems that may be linear or non-linear systems.

Approximation

$$u(x) = f_0 + \sum_{j=1}^{n} C_j f_j$$

where, C_j are constants to be determined. C_j are the generalized coordinates as the variation of them represents the variation of the displacement curve.

1. f_0 is chosen such that the non-homogeneous part of all essential boundary conditions are satisfied.
2. f_j are chosen such that the homogeneous part of all essential boundary conditions are satisfied. In other words, f_j are a set of kinematically admissible functions.
3. f_j must be sufficiently differentiable. In other words, the highest derivative in the functional Ω must be defined.
4. f_j must be independent. Otherwise, there exist values of C_j such that $\sum_{j=1}^{n} C_j f_j = 0$ and our matrix in the set of algebraic equations will be singular.
5. f_j must be complete. Otherwise, we may get large errors in our solution from our missing terms.
6. If all the above are satisfied then we have correctly chosen are functions, but the choice may not be a good choice because the series converges very slowly.

Algebraic equation

$$\Omega = \frac{1}{2}B(u_1,u_2) - l(u_1) \qquad u_1(x) = f_0 + \sum_{j=1}^{n} C_j f_j \quad \text{and} \quad u_2(x) = f_0 + \sum_{k=1}^{n} C_k f_k$$

$$l(u_1) = l(f_0 + \sum_{j=1}^{n} C_j f_j) = l(f_0) + \sum_{j=1}^{n} C_j l(f_j)$$

$$B(u_1,u_2) = B(f_0 + \sum_{j=1}^{n} C_j f_j, u_2) = B(f_0,u_2) + \sum_{j=1}^{n} C_j B(f_j,u_2) = B(f_0, f_0 + \sum_{k=1}^{n} C_k f_k) + \sum_{j=1}^{n} C_j B(f_j, f_0 + \sum_{k=1}^{n} C_k f_k) \text{ or}$$

$$B(u_1,u_2) = B(f_0,f_0) + \sum_{k=1}^{n} C_k B(f_0,f_k) + \sum_{j=1}^{n} C_j B(f_j,f_0) + \sum_{j=1}^{n} C_j \sum_{k=1}^{n} C_k B(f_j,f_k)$$

$$B(u_1,u_2) = B(f_0,f_0) + \sum_{j=1}^{n} C_j \{B(f_0,f_j) + B(f_j,f_0)\} + \sum_{j=1}^{n}\sum_{k=1}^{n} C_j C_k B(f_j,f_k)$$

$$\Omega = \left[\frac{1}{2}B(f_0,f_0) - l(f_0)\right] + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} C_j C_k B(f_j,f_k) - \sum_{j=1}^{n} C_j \left[l(f_j) - \frac{1}{2}\{B(f_0,f_j) + B(f_j,f_0)\}\right]$$

$$\Omega = \Omega_0 + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} C_j C_k B(f_j,f_k) - \sum_{j=1}^{n} C_j \left[l(f_j) - \frac{1}{2}\{B(f_0,f_j) + B(f_j,f_0)\}\right] \qquad \text{where} \qquad \Omega = \frac{1}{2}B(f_0,f_0) - l(f_0)$$

We take the first variation of potential energy and set it equal to zero to minimize the potential energy.

$$\delta\Omega = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} [\delta C_j C_k B(f_j,f_k) + C_j \delta C_k B(f_j,f_k)] - \sum_{j=1}^{n} \delta C_j \left\{ l(f_j) - \frac{1}{2}\{B(f_0,f_j) + B(f_j,f_0)\}\right\} \text{ or}$$

$$\delta\Omega = \frac{1}{2}\left\{\sum_{j=1}^{n}\sum_{k=1}^{n} [\delta C_j C_k B(f_j,f_k)] + \sum_{k=1}^{n}\sum_{j=1}^{n} [\delta C_j C_k B(f_k,f_j)]\right\} - \sum_{j=1}^{n} \delta C_j \left\{ l(f_j) - \frac{1}{2}\{B(f_0,f_j) + B(f_j,f_0)\}\right\}$$

Symmetric Bilinear functionals: $B(f_j,f_k) = B(f_k,f_j)$; $B(f_0,f_j) = B(f_j,f_0)$

$$\delta\Omega = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} [\delta C_j C_k \{B(f_j,f_k) + B(f_j,f_k)\}] - \sum_{j=1}^{n} \delta C_j \{l(f_j) - B(f_0,f_j)\} = \sum_{j=1}^{n} \delta C_j \left\{ \sum_{k=1}^{n} [B(f_j,f_k)C_k - \{l(f_j) - B(f_0,f_j)\}]\right\}$$

$$\delta\Omega = \sum_{j=1}^{n}\sum_{k=1}^{n} \delta C_j C_k B(f_j,f_k) - \sum_{j=1}^{n} \delta C_j [l(f_j) - B(f_0,f_j)] = \sum_{j=1}^{n}\sum_{k=1}^{n} \delta C_j [B(f_j,f_k)C_k - \{l(f_j) - B(f_0,f_j)\}] = 0 \text{ or}$$

$$\boxed{\sum_{k=1}^{n} B(f_j,f_k)C_k - \{l(f_j) - B(f_0,f_j)\} = 0 \qquad j = 1 \text{ to } n}$$

Matrix Form: $[K]\{C\} = \{R\}$ where $K_{jk} = B(f_j,f_k)$ $R_j = l(f_j) - B(f_0,f_j)$

$[K]$ is called the stiffness matrix and because the bilinear functional is symmetric, the stiffness matrix is symmetric. If $B(u, u) \geq 0$ for any u, then the stiffness matrix $[K]$ is positive definite—a property that is used in solution procedures of algebraic equations and in eigenvalue problems of dynamic systems. The convergence of the series is in the following sense

$$\Omega(u_N) \geq \Omega(u_M) \qquad \text{if} \qquad N \leq M$$

- As degree of freedom increases, the potential energy will decrease.

Potential Energy at equilibrium:

Let C_j^* represent the solution of the algebraic equations (the values at equilibrium), that is $\sum_{k=1}^{n} B(f_j, f_k) C_k^* = l(f_j) - B(f_0, f_j)$

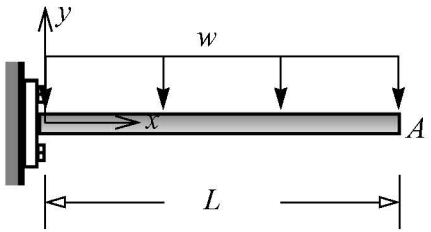
$$\Omega^* = \Omega_0 + \frac{1}{2}\sum_{j=1}^{n} C_j^* \left\{ \sum_{k=1}^{n} B(f_j, f_k) C_k^* \right\} - \sum_{j=1}^{n} C_j^* \{ l(f_j) - B(f_0, f_j) \} = \Omega_0 + \frac{1}{2}\sum_{j=1}^{n} C_j^* \{ l(f_j) - B(f_0, f_j) \} - \sum_{j=1}^{n} C_j^* \{ l(f_j) - B(f_0, f_j) \}$$

or

$$\Omega^* = \Omega_0 - \frac{1}{2}\sum_{j=1}^{n} C_j^* \{ l(f_j) - B(f_0, f_j) \} = \Omega_0 - \frac{1}{2}\sum_{j=1}^{n} C_j^* R_j \qquad \text{or} \qquad \Omega^* - \Omega_0 = -\frac{1}{2}\sum_{j=1}^{n} C_j^* R_j = -W^*/2$$

At equilibrium, the potential energy of the system is negative of half the work potential.

C.2 A beam and its loading are as shown below. Use the Rayleigh-Ritz method with one and two parameters to determine the deflection at $x = 0.25L$, $x = 0.5L$, $x = 0.75L$, and $x = L$, and the potential energy function. Compare your results with the analytical solution. Assume that EI is constant for the beam.



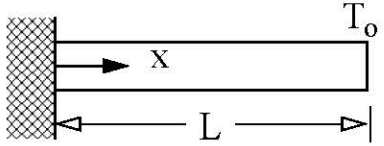
Overview of Approximate Methods

1-D Heat Conduction:

$$\frac{d^2T}{dx^2} = 0 \qquad 0 \le x \le L$$ ——— *Differential Equation*

$$-k\frac{dT}{dx}\bigg|_{x=0} = 0$$ ——— *Natural Boundary Condition*

$$T\big|_{x=L} = T_o$$ ——— *Essential Boundary Condition*

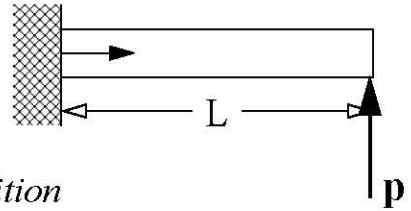


Beam Bending

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = p_y \qquad 0 \le x \le L$$ ——— *Differential Equation*

$$V = -\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = P \qquad M = EI\frac{d^2v}{dx^2} = 0$$ ——— *Natural Boundary Condition*

$$v(0) = 0 \qquad \frac{dv}{dx}\bigg|_{x=0} = 0$$ ——— *Essential Boundary Condition*

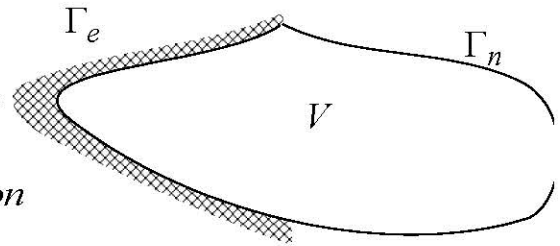


Approximation of Boundary Value Problem

$L\{u\} = h$ in V ——— *Differential Equation*

$D_n\{u\} = g_n$ on Γ_e ——— *Natural Boundary Condition*

$D_e\{u\} = g_e$ on Γ_n ——— *Essential Boundary Condition*



$u = f_o + \sum_{j=1}^{n} c_j f_j$

f_j set of approximating functions

set of f_j is complete and independent.

$e_d = \sum_{j=1}^{n} c_j L\{f_j\} - (h - L\{f_o\})$ ——— *Error in Differential Equation*

$e_n = \sum_{j=1}^{n} c_j D_n\{f_j\} - (g_n - D_n\{f_o\})$ ——— *Error in Natural Boundary Condition*

$e_e = \sum_{j=1}^{n} c_j D_e\{f_j\} - (g_e - D_e\{f_o\}_o)$ ——— *Error in Essential Boundary Condition*

Commonality and Differences in Approximate Methods

Commonalities

- *Produce a set of algebraic equations in the unknown constants c_j.*
- *Choose f_i to set one (or two) of the errors e_d, e_e, or e_n to zero*
- *Minimize the remaining error.*

Differences

- *Which error is set to zero*

  Domain Methods: $e_e = 0$ or $e_n = 0$

  Boundary Methods: $e_d = 0$

- *Error Minimizing Process*

Error Minimization

$$\int_V \psi_i^{(d)} e_d dV + \int_{\Gamma_e} \psi_i^{(e)} e_e ds + \int_{\Gamma_n} \psi_i^{(n)} e_n ds = 0$$ ——— Weighted Residue

FEM-Stiffness version: $e_e = 0$

$$\int_V \psi_i^{(d)} e_d d\Omega + \int_{\Gamma_n} \psi_i^{(n)} e_n ds = 0$$

FEM-Flexibility version: $e_n = 0$

$$\int_V \psi_i^{(d)} e_d d\Omega + \int_{\Gamma_e} \psi_i^{(e)} e_e ds = 0$$

BEM: $e_d = 0$

$$\int_{\Gamma_n} \psi_i^{(n)} e_n ds + \int_{\Gamma_e} \psi_i^{(e)} e_e ds = 0$$

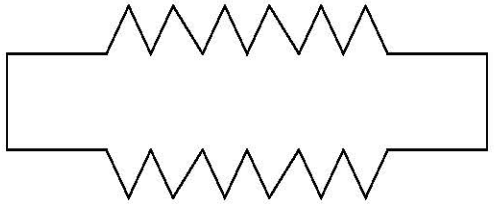
FEM: Discretization process is on domain of the entire body V

BEM: Discretization process is on the boundary of the body Γ

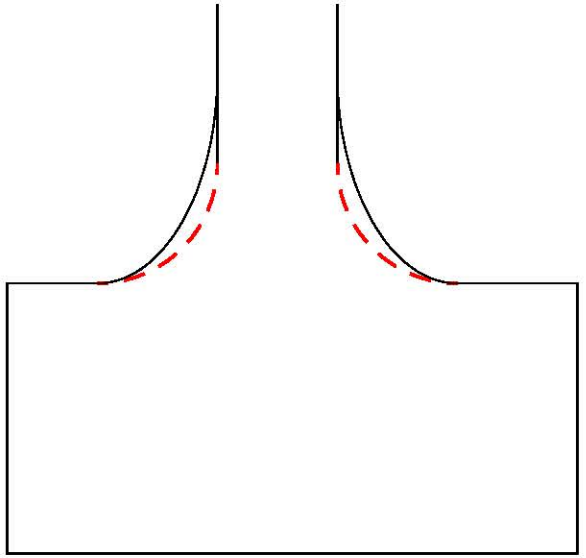
Class of Problems for BEM

I. Complex Boundary Shapes.

Grain Boundaries
Geological Flaws
Threads in a screw



II. Shape Optimization of Components.



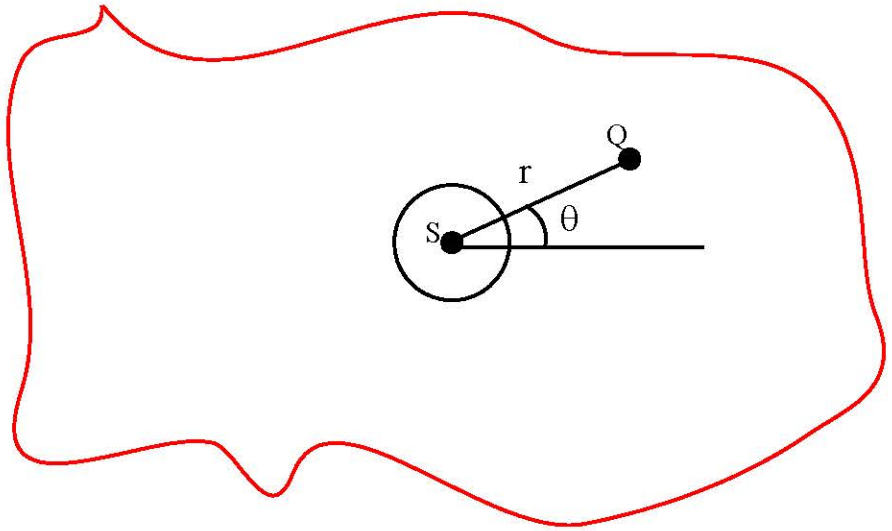
Influence Functions (Fundamental Solution)

Have to find ϕ_i such that $e_D = 0$

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$-k\frac{\partial T}{\partial n} \to 0 \qquad as \qquad Q \to \infty$$

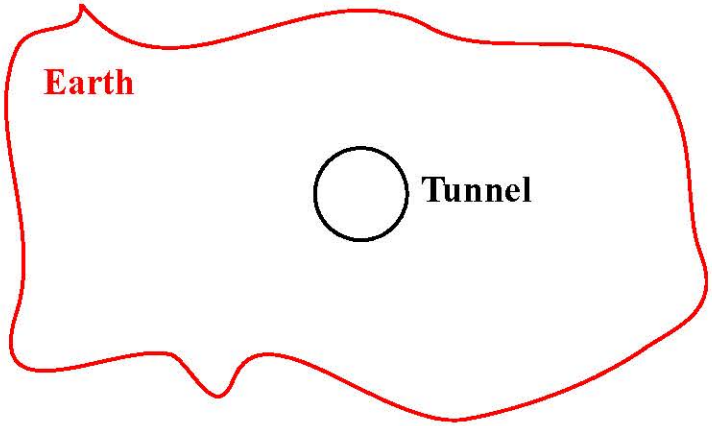
$$T < \infty \qquad as \qquad Q \to \infty$$

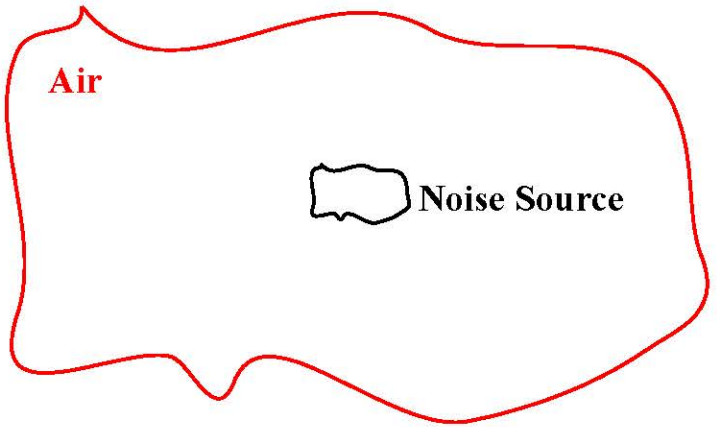


1. Conditions at infinity are implicitly satisfied in the formulation.
2. Influence Functions are singular at source point.
3. Influence Functions are not unique.

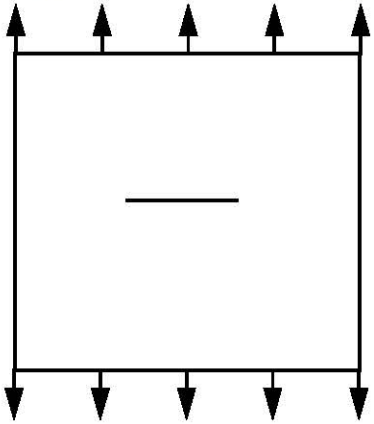
Class of Problems for BEM

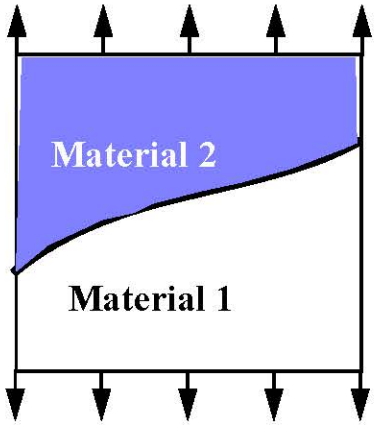
III. Infinite Domain





IV. Large Gradients (near singularities)





Other Methods for Error Minimization

The methods below can be applied to boundary methods also. We will restrict our discussion to domain methods.

The error is: $e_d = \sum_{j=1}^{n} c_j L\{f_j\} - (h - L\{f_o\}) = \sum_{j=1}^{n} c_j \gamma_j + H$ where $\gamma_j = L\{f_j\}; H = -(h - L\{f_o\})$

Objective is to obtain: $\sum_{j=1}^{n} K_{kj} c_j = R_k$

1. Weighted Residue (Petro-Galerkin)

$$\int_V \psi_k e_d dV = \int_V \psi_k [\sum_{j=1}^{n} c_j \gamma_j + H] dV = \int_V \psi_k H dV + \sum_{j=1}^{n} c_j \int_V \psi_k \gamma_j dV = 0$$

$$\int_V \psi_k H dV = -R_k \qquad \int_V \psi_k \gamma_j dV = K_{kj}$$

Makes the error orthogonal (minimizes) to the weighting functions ψ_k. Very general but depends upon the choice of ψ_k

2. Collocation

$$e_d(x_k) = H(x_k) + \sum_{j=1}^{n} c_j \gamma_j(x_k) = 0$$

$$H(x_k) = -R_k \qquad \gamma_j(x_k) = K_{kj}$$

Simplest method. General but depends upon location of x_k

3. Least Square

$$E = \int_V e_d^2 dV \qquad \frac{\partial E}{\partial c_k} = 0 \qquad or \qquad \frac{\partial E}{\partial c_k} = 2\int_V \left(\frac{\partial e_d}{\partial c_k} e_d\right) dV = 2\int_\Omega \gamma_k e_d dV = 0 \qquad \text{or}$$

$$\int_V [H + \sum_{j=1}^{n} c_j \gamma_j]\gamma_k dV = \underbrace{\int_V H\gamma_k dV}_{-R_k} + \sum_{j=1}^{n} c_j \underbrace{\int_V \gamma_j \gamma_k dV}_{K_{kj}} = 0$$

It is general. Minimizes error. Produces a symmetric matrix. Can be computationally very expensive.

4. Galerkin

In weighted residue choose: $\psi_k = f_k$

$$\int_V f_k e_d dV = \int_V f_k [H + \sum_{j=1}^{n} c_j \gamma_j] dV = \underbrace{\int_V f_k h dV}_{-R_k} + \sum_{j=1}^{n} c_j \underbrace{\int_V f_k \gamma_j dV}_{K_{kj}} = 0$$

General and will produce same results as Rayleigh Ritz's for problems that have functionals of the type needed by Rayleigh-Ritz method.

C.3 A boundary value problem is as shown below

$$\frac{d^2u}{dx^2} + u = x^2 \qquad u(0) = 0 \qquad u(1) = 0$$

Find 1 parameter solution and compare it for the following methods.

(a) Rayleigh-Ritz method. Use the following functional: $\Omega = \frac{1}{2}\int_0^1 \left[\left(\frac{du}{dx}\right)^2 - u^2\right]dx + \int_0^1 x^2 u\,dx$

(b) Collocation method. Use midway point to set error to zero.

(c) Least square method.

(d) Weighted residue. Use $\psi_1 = x$

(e) Galerkin's method.

Weak Form

In many engineering application we have the boundary value problem but not a functional like potential energy. We will first go through the steps of obtaining the boundary value problem from a functional for axial members, then we will reverse it to understand the basis of weak form. Then develop a procedure for developing the weak form from a given boundary value problem.

Prelude to weak form

Forward Process.

Obtain the boundary value problem for axial displacement $u(x)$ by minimizing the potential energy below

$$\Omega = \frac{1}{2}\int_L EA\left(\frac{du}{dx}\right)^2 dx - \int_L p_x(x)u(x)dx - Pu(L)$$

where, L is the length of the beam; E is modulus of elasticity; A is the cross sectional area; p_x is distributed axial force per unit length; and P is a point force applied at $x = L$.

Step 1 First variation

$$\delta\Omega = \int_L EA\frac{du}{dx}\delta\left(\frac{du}{dx}\right) dx - \int_L p_x(x)\delta u(x)dx - P\delta u(L)$$

Step 2 Integration by parts

$$\delta\Omega = EA\frac{du}{dx}\delta u\Big|_0^L - \int_L \frac{d}{dx}\left(EA\frac{du}{dx}\right)\delta u\, dx - \int_L p_x(x)\delta u(x)dx - P\delta u(L) \text{ or}$$

$$\delta\Omega = \left[EA\frac{du}{dx}(L) - P\right]\delta u(L) - \left[EA\frac{du}{dx}(0)\right]\delta u(0) - \int_L \left[\frac{d}{dx}\left(EA\frac{du}{dx}\right) + p_x(x)\right]\delta u(x)\, dx$$

Step 3 Boundary value problem: $\delta\Omega = 0$

$$\text{Differential Equation: } \frac{d}{dx}\left(EA\frac{du}{dx}\right) + p_x(x) = 0 \qquad 0 \le x \le L$$

$$\text{Boundary Conditions: } u(0) = u_0 \qquad EA\frac{du}{dx}(L) = P$$

Reverse Process.
Obtain the functional from the given boundary value problem

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + p_x(x) = 0 \qquad 0 \le x \le L;\ u(0) = u_0 \qquad EA\frac{du}{dx}(L) = P$$

Step 1 Multiply the differential equation by $-\delta u$ and integrate from 0 to L.

$$\delta\Omega = -\int_L \frac{d}{dx}\left(EA\frac{du}{dx}\right)\delta u\, dx - \int_L p_x \delta u\, dx$$

Step 2 Integrate by parts and transfer **half** of the derivatives. If the order of differential equation is $2r$ then transfer r derivatives.

$$\delta\Omega = -EA\frac{du}{dx}\delta u\Big|_0^L + \int_L \left(EA\frac{du}{dx}\right)\frac{d}{dx}(\delta u)\, dx - \int_L p_x(x)\delta u(x)dx \text{ or}$$

$$\delta\Omega = -EA\frac{du}{dx}(L)\delta u(L) + EA\frac{du}{dx}(0)\delta u(0) + \int_L \left(EA\frac{du}{dx}\right)\frac{d}{dx}(\delta u)\, dx - \int_L p_x(x)\delta u(x)dx$$

Step 3 Substitute boundary conditions. Note $u(0) = u_0$ implies $\delta u(0) = 0$

$$\delta\Omega = -P\delta u(L) + \int_L \left(EA\frac{du}{dx}\right)\frac{d}{dx}(\delta u)\, dx - \int_L p_x(x)\delta u(x)dx \text{ or}$$

$$\delta\Omega = -P\mathrm{v}(L) + \int_L EA\frac{du}{dx}\frac{d\mathrm{v}}{dx}dx - \int_L p_x(x)\mathrm{v}(x)dx \qquad \text{where } \mathrm{v} = \delta u$$

Step 4 Identify the bilinear and linear functionals.

$$B(u,\mathrm{v}) = \int_L EA\frac{du}{dx}\frac{d\mathrm{v}}{dx}dx \qquad l(\mathrm{v}) = \int_L p_x(x)\mathrm{v}(x)dx + P\mathrm{v}(L)$$

Step 5 IF bilinear functional is symmetric, that is, $B(u,\mathrm{v}) = B(\mathrm{v},u)$ then

$$\Omega = \frac{1}{2}B(u,u) - l(u) \text{ [See Reddy]}$$

$$\Omega = \frac{1}{2}\int_L EA\left(\frac{du}{dx}\right)^2 dx - \int_L p_x u\, dx - Pu(L)$$

Procedure to develop weak from BVP

Procedure is valid for 1-D, 2-D, and 3-D.
We will assume the differential equation is of order $2r$. Note v is δu in the procedure below.

Step 1 Multiply the differential equation with $(-1)^r \mathrm{v}$ and integrate over the region. Note v (δu) must satisfy the homogeneous essential boundary conditions and the principal derivatives up to order r) must be continuous.

Step 2 Transfer half the derivatives from u on to v by parts (Green's formula, Divergence theorem). Thus, if the oder of differential equation is $2r$ then transfer r derivatives.

Step 3 Substitute all the natural boundary conditions on u.

Step 4 Substitute all the essential homogeneous boundary condition on v (δu). Some terms will drop out of the equation.

Step 5 Collect all the terms in which only v appears (and not u). Label these by the linear operator $l(\mathrm{v})$.

Step 6 Collect all terms in which u and v appear. Label these by the bilinear operator $B(\mathrm{v}, u)$

Step 7 The integral expression is $\delta\Omega = B(\mathrm{v}, u) - l(\mathrm{v}) = B(\delta u, u) - l(\delta u)$

We will use the formulation of Step 7 in finite element method.

- If there are more than 1 differential equation then in Step 1 we will multiply equation 1 by $(-1)^{r_1}\mathrm{v}_1$, equation 2 by $(-1)^{r_2}\mathrm{v}_2$ and add the product to define $\delta\Omega$. We then proceed as above for each term.

C.4 Construct the weak form for the following boundary value problem

$$\frac{d^2u}{dx^2} + u = x^2 \qquad 0 \le x \le 1\,;\; u(0) = u_0 \qquad \frac{du}{dx}(1) = 1$$