Variational Calculus in 3-D

Gauss divergence formula is given by the equation below.

$$\iiint_T \left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right] dV = \iint_S [u_x n_x + u_y n_y + u_z n_z] dA$$

where, u_x, u_y, and u_z are the components of a vector function which is continuous with continuous first derivatives in a region T bounded by a smooth surface S; and n_x, n_y, and n_z are the direction cosines of a unit normal on surface S.

Let each of the components of u be equal to product of two continuous functions fg while the other two components are zero. This produces the formulas below.

$$\iiint_T \frac{\partial f}{\partial x} g\, dV = \iint_S fg n_x\, dA - \iiint_T f \frac{\partial g}{\partial x} dV \quad \text{and} \quad \iiint_T \frac{\partial f}{\partial y} g\, dV = \iint_S fg n_y\, dA - \iiint_T f \frac{\partial g}{\partial y} dV \quad \text{and} \quad \iiint_T \frac{\partial f}{\partial z} g\, dV = \iint_S fg n_z\, dA - \iiint_T f \frac{\partial g}{\partial z} dV$$

Stationary value of a definite volume integral

$$I(u) = \iiint_T H(u_{,x}, u_{,y}, u_{,z}, u, x, y, z)\, dx\, dy\, dz$$

Differential Equation: $\boxed{\frac{\partial H}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u_{,x}}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial u_{,y}}\right) - \frac{\partial}{\partial z}\left(\frac{\partial H}{\partial u_{,z}}\right) = 0 \qquad x, y, z \text{ in } T}$

Boundary Conditions: $\boxed{\left(\frac{\partial H}{\partial u_{,x}}\right) n_x + \left(\frac{\partial H}{\partial u_{,y}}\right) n_y + \left(\frac{\partial H}{\partial u_{,z}}\right) n_z = 0 \quad \text{or} \quad \delta u = 0 \qquad x, y, z \text{ on S}}$

C.1 Obtain the boundary value problem for an elastic body in three dimensions with only body forces. The potential energy of the elastic body is

$$\Omega(u, v, w) = G\iiint_T\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \frac{\nu}{1-2\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2\right]dx\,dy\,dz$$

$$-\iiint_T [F_x u + F_y v + F_z w]\,dx\,dy\,dz$$

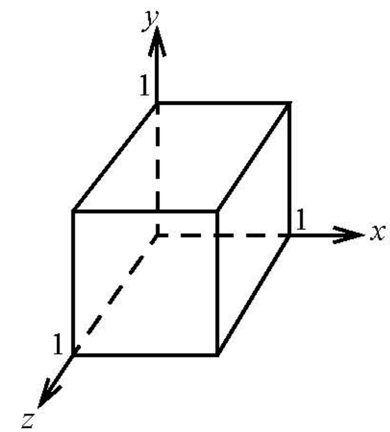
where, u, v, and w are the displacement in the x, y, *and* z directions; F_x, F_y, and F_z are the body forces per unit volume in the x, y, *and* z directions; G and ν are the shear modulus of elasticity and Poisson's ratio.

C 2 Obtain the weak form for the following boundary value problem and identify the bilinear and linear functional.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = -1$$

$$T = T_o \qquad \text{on } x = 1;\ y = 1;\ \text{and } z = 1$$

$$\frac{\partial T}{\partial n} = 1 \qquad \text{on } x = 0;\ y = 0;\ \text{and } z = 0$$



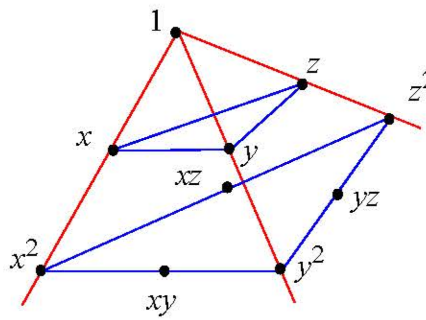
3-D Elements

Our Reference 5: T.J. Chung "Finite element analysis in fluid dynamics" McGraw-Hill Book Company.New York

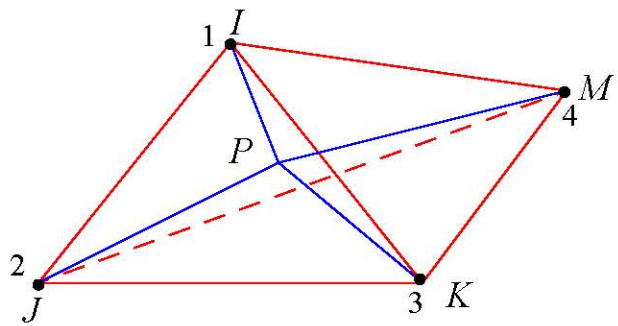
Tetrahedron

Linear Approximation: $u = a_o + a_1 x + a_2 y + a_3 z$

4 unknown constants, need 4 nodes---Tetrahedron.



Volume Coordinates



$$L_I = \frac{\text{Volume } PJKM}{V} = \frac{V_I}{V}$$

$$L_J = \frac{\text{Volume } PKMI}{V} = \frac{V_J}{V}$$

$$L_K = \frac{\text{Volume } PMIJ}{V} = \frac{V_K}{V}$$

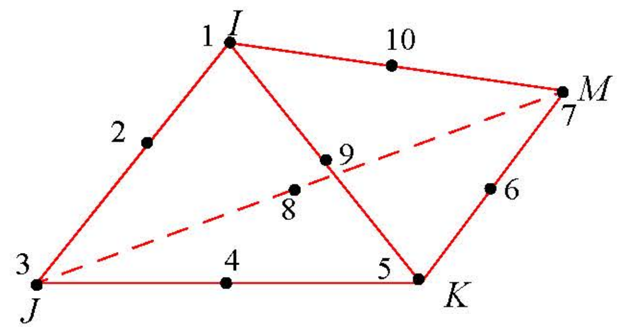
$$L_M = \frac{\text{Volume } PIJK}{V} = \frac{V_M}{V}$$

V = Volume of the tetrahedron IJKM=$V_I+V_J+V_K+V_M$

$$L_I + L_J + L_K + L_M = 1$$

Linear Element: $\psi_1 = L_I \qquad \psi_2 = L_J \qquad \psi_3 = L_K \qquad \psi_4 = L_M$

Quadratic Element:

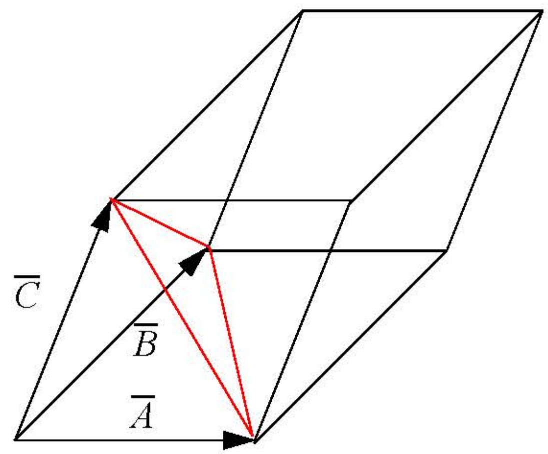


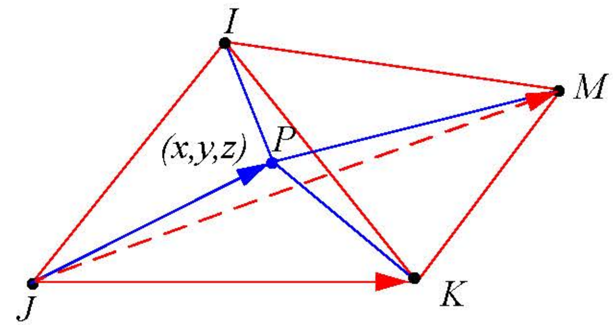
$$\psi_1 = \frac{L_I\left(L_I - \frac{1}{2}\right)}{(1)\left(1 - \frac{1}{2}\right)} = 2L_I\left(L_I - \frac{1}{2}\right)$$

$$\psi_2 = \frac{L_I L_J}{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = 4L_I L_J$$

Write ψ_4 and ψ_5

Coordinate Relation





volume of cube $= \overline{C} \bullet (\overline{A} \times \overline{B})$

volume of tetrahedron $= \frac{1}{6}[\overline{C} \bullet (\overline{A} \times \overline{B})]$

$$L_I = \frac{1}{6V}[\overline{JP} \bullet (\overline{JK} \times \overline{JM})]$$

$$\overline{JP} = (x - x_J)\bar{i} + (y - y_J)\bar{j} + (z - z_J)\bar{k}$$

$$\overline{JK} = (x_K - x_J)\bar{i} + (y_K - y_J)\bar{j} + (z_K - z_J)\bar{k} = x_{KJ}\bar{i} + y_{KJ}\bar{j} + z_{KJ}\bar{k}$$

$$\overline{JM} = (x_M - x_J)\bar{i} + (y_M - y_J)\bar{j} + (z_M - z_J)\bar{k} = x_{MJ}i + y_{MJ}\bar{j} + z_{MJ}\bar{k}$$

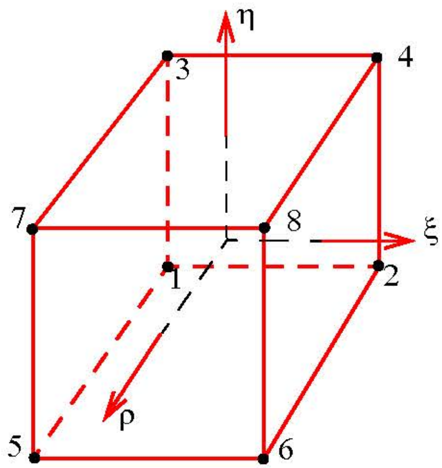
$$L_I = \frac{1}{6V}\begin{bmatrix} (x - x_J)(y_{KJ}z_{MJ} - z_{KJ}y_{MJ}) \\ +(y - y_J)(z_{KJ}x_{MJ} - x_{MJ}z_{KJ}) \\ +(z - z_J)(x_{KJ}y_{MJ} - y_{KJ}x_{MJ}) \end{bmatrix}$$

$$\frac{\partial \psi_i}{\partial x} = \frac{\partial \psi_i \partial L_I}{\partial L_I \partial x} + \frac{\partial \psi_i \partial L_J}{\partial L_J \partial x} + \frac{\partial \psi_i\, \partial L_K}{\partial L_K \partial x} + \frac{\partial \psi_i\, \partial L_M}{\partial L_M \partial x}$$

$$\iiint_V L_I^m L_J^n L_K^p L_M^q \; dx\,dy\,dz = (6V)\frac{m!\,n!\,p!\,q!}{(m+n+p+q+3)!}$$

Brick Elements (Hexahedron)

Linear

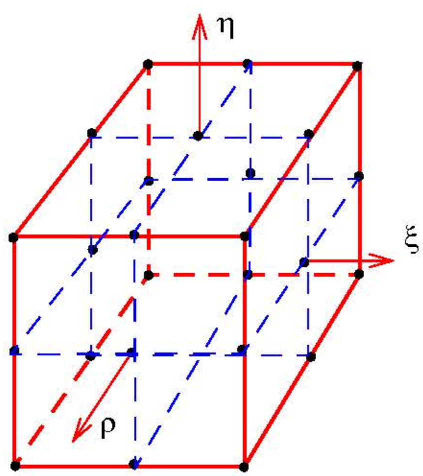


$$\psi_1 = \frac{1}{8}(1-\xi)(1-\eta)(1-\rho)$$

$$\psi_2 = \frac{1}{8}(1+\xi)(1-\eta)(1-\rho)$$

Write ψ_4 and ψ_5.

Quadratic

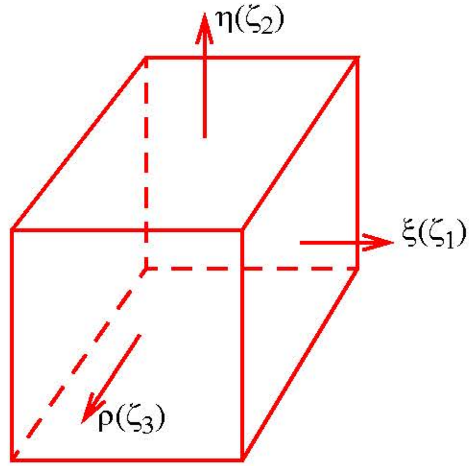


27 nodes

$\psi_i = \phi_1(\xi)\phi_2(\eta)\phi_3(\rho)$ where the ϕ's are chosen from 1-D.

Serendipity Elements:

No nodes either on the surface or in the interior. The nodes are only along the edges.



Linear:

$$u = a + \sum_{j=1}^{3} b_j\zeta_j + \sum_{k=1}^{3}\sum_{j=1}^{3} c_{jk}\zeta_j\zeta_k + \sum_{l=1}^{3}\sum_{k=1}^{3}\sum_{j=1}^{3} d_{jkl}\zeta_j\zeta_k\zeta_l$$

where $c_{jk} = 0$ *and* $d_{jkl} = 0$ anytime two indicies are equal

The above approximation is linear along any line.

Quadratic:

$$u = a + \sum_{j=1}^{3} b_j\zeta_j + \sum_{k=1}^{3}\sum_{j=1}^{3} c_{jk}\zeta_j\zeta_k + \sum_{l=1}^{3}\sum_{k=1}^{3}\sum_{j=1}^{3} d_{jkl}\zeta_j\zeta_k\zeta_l + \sum_{jm=1}^{3}\sum_{l=1}^{3}\sum_{k=1}^{3}\sum_{j=1}^{3} e_{jklm}\zeta_j\zeta_k\zeta_l\zeta_m$$

where $d_{jkl} = 0$ *and* $e_{jklm} = 0$ anytime two indicies are equal

No condition on c_{jk}.

The above approximation is quadratic along any line.