Variational Calculus 2-D

The learning objectives in this chapter are:

- Understand the application of variational calculus to obtain boundary value problems in two dimensions.

Stationary value of a definite area integral

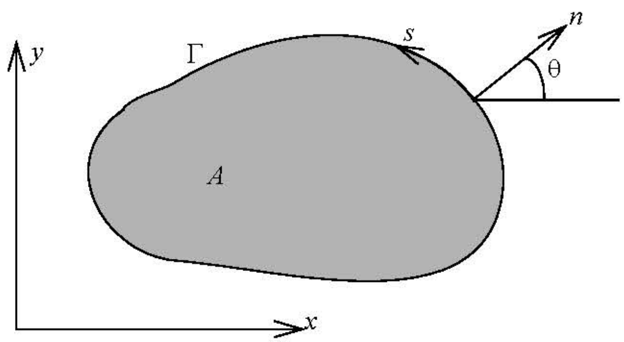
- In stationary value of an area integral the transfer of derivative is accomplished by using Green's theorem.

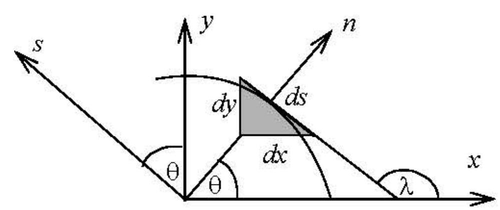
Mathematical preliminaries

The Green's formulas is written below for convenience.

$$\iint_A f\frac{\partial g}{\partial x}dx\,dy = \oint_\Gamma fg\,dy - \iint_A g\frac{\partial f}{\partial x}dx\,dy \qquad \iint_A f\frac{\partial g}{\partial y}dx\,dy = -\oint_\Gamma fg\,dx - \iint_A g\frac{\partial f}{\partial y}dx\,dy$$

where $f(x, y)$ and $g(x, y)$ are two continuos functions in the area A that is bounded by the curve Γ.





(b)

We can relate the Cartesian coordinates (*x*, *y*) and to normal and tangential coordinates (*n*, *s*).

$$x = n\cos\theta - s\sin\theta \qquad y = n\sin\theta + s\cos\theta$$

$$n = x\cos\theta + y\sin\theta \qquad s = -x\sin\theta + y\cos\theta$$

We define the direction cosines of the unit normal as

$$n_x = \frac{\partial n}{\partial x} = cos\theta = \frac{\partial s}{\partial y} \qquad n_y = \frac{\partial n}{\partial y} = sin\ \theta = -\frac{\partial s}{\partial x}$$

If a point is restricted to the boundary, then $dn = 0$

$$dx = (-sin\theta)ds = -n_y ds \qquad dy = (cos\theta)ds = n_x ds$$

$$\iint_A f\frac{\partial g}{\partial x}dxdy = \oint_\Gamma (fg)n_x ds - \iint_A g\frac{\partial f}{\partial x}dxdy \qquad \iint_A f\frac{\partial g}{\partial y}dxdy = \oint_\Gamma (fg)n_y dy - \iint_A g\frac{\partial f}{\partial y}dxdy$$

By Chain rule: $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial n}\frac{\partial n}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial f}{\partial n}n_x - \frac{\partial f}{\partial s}n_y \qquad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial n}\frac{\partial n}{\partial y} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial y} = \frac{\partial f}{\partial n}n_y + \frac{\partial f}{\partial s}n_x$

$$\frac{\partial n_x}{\partial s} = -sin\theta\left(\frac{\partial\theta}{\partial s}\right) = -n_y\left(\frac{\partial\theta}{\partial s}\right) = -\left(\frac{n_y}{R_{cur}}\right) \qquad \frac{\partial n_y}{\partial s} = cos\theta\left(\frac{\partial\theta}{\partial s}\right) = n_x\left(\frac{\partial\theta}{\partial s}\right) = \frac{n_x}{R_{cur}}$$

where, R_{cur} is the radius of curvature of the boundary at the point under consideration and can be function of s.

Stationary value of a functional with first order derivatives

$$I(u) = \iint_A H(u_{,x}, u_{,y}, u, x, y)dxdy \qquad \text{where} \qquad u_{,x} = \partial u/\partial x \qquad u_{,y} = \partial u/\partial y$$

- *A subscript without a comma implies a component and with a comma implies derivative.*

$$= \iint_A \left[\left(\frac{\partial H}{\partial u_{,x}}\right)\delta u_x + \left(\frac{\partial H}{\partial u_{,y}}\right)\delta u_y + \left(\frac{\partial H}{\partial u}\right)\delta u\right]dxdy = \iint_A \left[\left(\frac{\partial H}{\partial u_{,x}}\right)\frac{\partial}{\partial x}(\delta u) + \left(\frac{\partial H}{\partial u_{,y}}\right)\frac{\partial}{\partial y}(\delta u) + \left(\frac{\partial H}{\partial u}\right)\delta u\right.$$

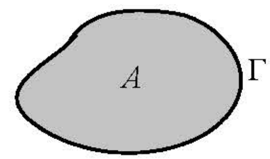
$$\delta I(u) = \oint_\Gamma \left[\frac{\partial H}{\partial u_{,x}}n_x + \left(\frac{\partial H}{\partial u_{,y}}\right)n_y\right]\delta u ds + \iint_A \left[\left(\frac{\partial H}{\partial u}\right) - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u_{,x}}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial u_{,y}}\right)\right]\delta u dxdy = 0$$

Differential Equation: $\boxed{\frac{\partial H}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u_{,x}}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial u_{,y}}\right) = 0 \qquad x, y \text{ in } R}$

Boundary Conditions: $\boxed{\left(\frac{\partial H}{\partial u_{,x}}\right)n_x + \left(\frac{\partial H}{\partial u_{,y}}\right)n_y = 0 \qquad \text{or} \qquad \delta u = 0 \qquad x, y \text{ on } S}$

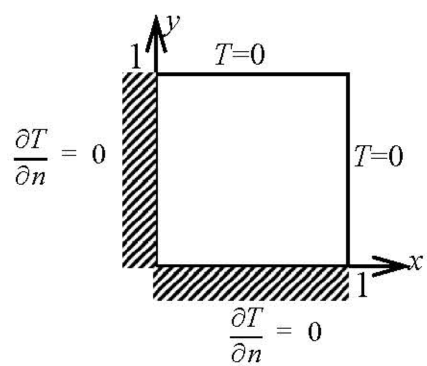
C.1 (a) In heat conduction, the thermal energy of a plate is given by the functional below. Obtain the boundary value problem.

$$I = \frac{1}{2}\iint_A \left\{ k\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right] - 2T \right\} dx\,dy$$



Ritz's approximation $T(x, y) = \phi_o(x, y) + \sum_i \sum_j C_{ij}\phi_{ij}(x, y)$ [See Reddy example 2.5.3]

(b) Find 1 parameter solution for the 1x1 square shown.

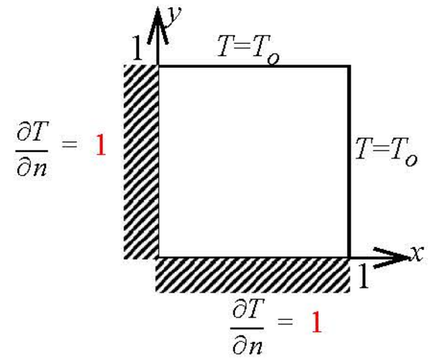


C.2 Obtain the weak form for the following boundary value problem.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -1$$

$$T = T_o \qquad \text{on } x = 1 \text{ and } y = 1$$

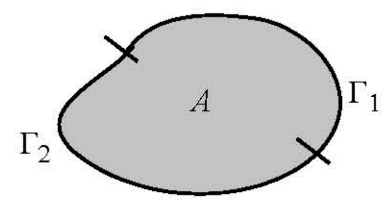
$$\frac{\partial T}{\partial n} = 1 \qquad \text{on } x = 0 \text{ and } y = 0$$



C.3 (a) Obtain the weak form of the given elastostatic boundary value problem for isotropic materials in plane strain. (b) Construct the functional whose stationary value will give us the boundary value problem.

$$\frac{\partial}{\partial x}\left[(2\mu+\lambda)\frac{\partial u}{\partial x}+\lambda\frac{\partial v}{\partial y}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+f_x = 0$$

$$\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\lambda\frac{\partial u}{\partial x}+(2\mu+\lambda)\frac{\partial v}{\partial y}\right]+f_y = 0 \qquad x, y \text{ in } A$$



$$\left[(2\mu+\lambda)\frac{\partial u}{\partial x}+\lambda\frac{\partial v}{\partial y}\right]n_x+\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)n_y = t_x$$

$$\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]n_x+\left[\lambda\frac{\partial u}{\partial x}+(2\mu+\lambda)\frac{\partial v}{\partial y}\right]n_y = t_y \qquad x, y \text{ on } \Gamma_1$$

$$u = u_o \qquad v = v_o \qquad x, y \text{ on } \Gamma_2$$

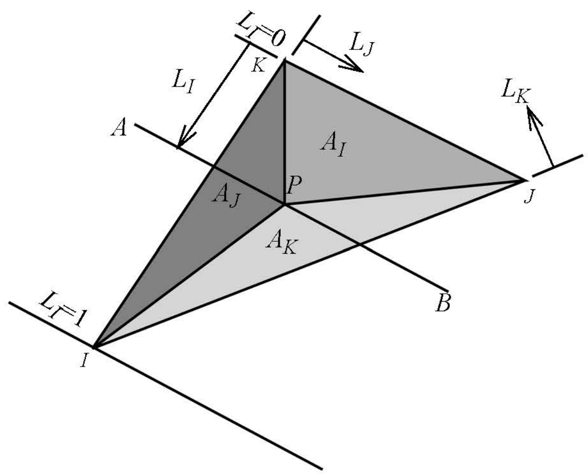
where, u and v are the components of displacements in x and y direction; f_x and f_y are the components of body force and t_x and t_y are the components of applied tractions; u_o and v_o are the specified displacements; and μ and λ are the shear modulus and Lame's constants.

Triangular Element For Second Order System in 2-D

Linear: $u(x) = C_0 + C_1 x + C_2 y$

The triangular element is the simplest element that can be used for modeling regions with curved boundary.

- **Area coordinates** are natural coordinates used in triangular element.



$$L_I = \frac{A_I}{A} \qquad L_J = \frac{A_J}{A} \qquad L_K = \frac{A_K}{A} \qquad L_I + L_J + L_K = 1$$

$$|\overline{PJ} \otimes \overline{PK}| = 2(\text{Area of triangle } PJK)$$

$$A_I = [(x_J - x)(y_K - y) - (x_K - x)(y_J - y)]/2$$

$$A_J = [(x_K - x)(y_I - y) - (x_I - x)(y_K - y)]/2$$

$$A_K = [(x_I - x)(y_J - y) - (x_J - x)(y_I - y)]/2$$

$$A = [(x_J - x_I)(y_K - y_I) - (x_K - x_I)(y_J - y_I)]/2$$

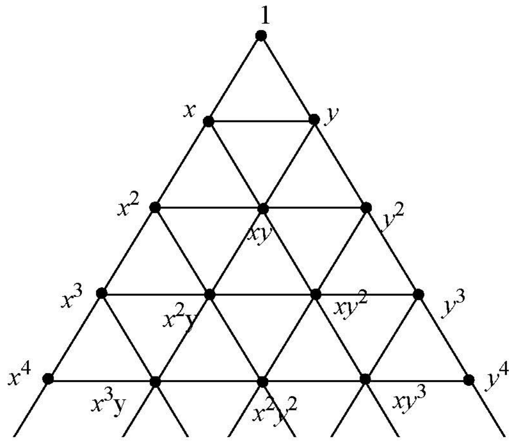
$$\frac{\partial L_I}{\partial x} = \frac{y_{JK}}{2A} \qquad \frac{\partial L_I}{\partial y} = \frac{-x_{JK}}{2A} \qquad \text{where} \qquad x_{JK} = x_J - x_K \text{ and } y_{JK} = y_J - y_K$$

$$\frac{\partial L_J}{\partial x} = \frac{y_{KI}}{2A} \qquad \frac{\partial L_J}{\partial y} = \frac{-x_{KI}}{2A} \qquad \text{where} \qquad x_{KI} = x_K - x_I \text{ and } y_{KI} = y_K - y_I$$

$$\frac{\partial L_K}{\partial x} = \frac{y_{IJ}}{2A} \qquad \frac{\partial L_K}{\partial y} = \frac{-x_{IJ}}{2A} \qquad \text{where} \qquad x_{IJ} = x_I - x_J \text{ and } y_{IJ} = y_I - y_J$$

Pascal's Triangle

Pascal's triangle helps determine how many nodes should be on an element to get a particular order of polynomial.



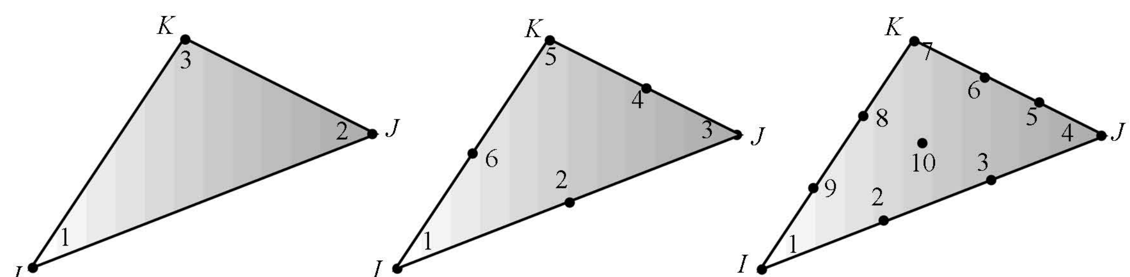
Lagrange Polynomials

Linear $\mathcal{L}_1 = L_I \quad \mathcal{L}_2 = L_J \quad \mathcal{L}_3 = L_K$

Quadratic

$$\mathcal{L}_1 = L_I(2L_I - 1) \quad \mathcal{L}_3 = L_J(2L_J - 1) \quad \mathcal{L}_5 = L_K(2L_K - 1)$$

$$\mathcal{L}_2 = 4L_IL_J \quad \mathcal{L}_4 = 4L_JL_K \quad \mathcal{L}_6 = 4L_KL_I$$



Linear Quadratic Cubic

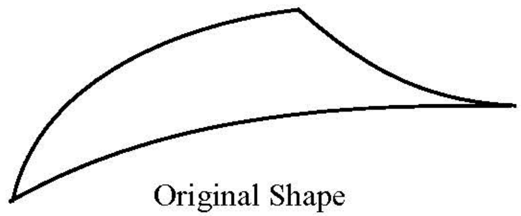
$$\iint_A L_I^m L_J^n L_K^p dx dy = (2A)\frac{m!n!p!}{(m+n+p+2)!} \qquad \int_a^b L_I^m L_J^n L_K^p \, ds = (b-a)\frac{m!n!p!}{(m+n+p+1)!}$$

C.4 Obtain the cubic Lagrange interpolation function for triangular elements for nodes 7, 8, and 9.

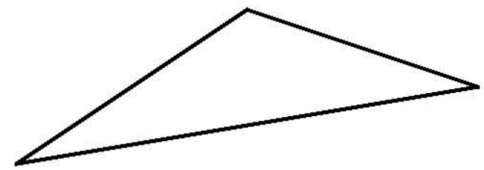
C.5 Obtain the formula for I_1 and I_2 in Appendix A.

C.6 Obtain $K_{11}^{(e)}$ and $K_{12}^{(e)}$ of the stiffness matrix and $R_1^{(e)}$ of the right hand side vector for Poisson's equation with quadratic triangular element. [See Appendices].

Isoparametric Formulation



Original Shape



Transformed Shape

Primary variable approximation: $u = \sum_{i=1}^{M} u_i^{(e)} \phi_i(L_I, L_J, L_K)$

Coordinate Transformation: $x = \sum_{i=1}^{N} x_i^{(e)} \psi_i(L_I, L_J, L_K)$ $\quad y = \sum_{i=1}^{N} y_i^{(e)} \psi_i(L_I, L_J, L_K)$ $\quad \psi_i = \mathcal{L}_1$

Sub-parametric: $M < N$; Isoparametric: $M = N$ Super-parametric: $M > N$ ---Not used

$$dx = \left(\sum_{i=1}^{N} x_i^{(e)} \frac{\partial \psi_i}{\partial L_I}\right) dL_I + \left(\sum_{i=1}^{N} x_i^{(e)} \frac{\partial \psi_i}{\partial L_J}\right) dL_J + \left(\sum_{i=1}^{N} x_i^{(e)} \frac{\partial \psi_i}{\partial L_K}\right) dL_K$$

$$dy = \left(\sum_{i=1}^{N} y_i^{(e)} \frac{\partial \psi_i}{\partial L_I}\right) dL_I + \left(\sum_{i=1}^{N} y_i^{(e)} \frac{\partial \psi_i}{\partial L_J}\right) dL_J + \left(\sum_{i=1}^{N} y_i^{(e)} \frac{\partial \psi_i}{\partial L_K}\right) dL_K$$

$$\begin{Bmatrix} dx \\ dy \end{Bmatrix} = [J] \begin{Bmatrix} dL_I \\ dL_J \\ dL_K \end{Bmatrix} \qquad [J] = \begin{bmatrix} \left(\sum_{i=1}^{N} x_i^{(e)} \frac{\partial \psi_i}{\partial L_I}\right) & \left(\sum_{i=1}^{N} x_i^{(e)} \frac{\partial \psi_i}{\partial L_J}\right) & \left(\sum_{i=1}^{N} x_i^{(e)} \frac{\partial \psi_i}{\partial L_K}\right) \\ \left(\sum_{i=1}^{N} y_i^{(e)} \frac{\partial \psi_i}{\partial L_I}\right) & \left(\sum_{i=1}^{N} y_i^{(e)} \frac{\partial \psi_i}{\partial L_J}\right) & \sum_{i=1}^{N} y_i^{(e)} \frac{\partial \psi_i}{\partial L_K} \end{bmatrix}$$ ----Jacobian matrix

- The Jacobian matrix is a rectangular matrix and cannot be inverted. But $L_I + L_J + L_K = 1$ or $dL_I + dL_J + dL_K = 0$. We can eliminate one of the three area coordinates and then proceed. See Zienkiewicz (Ref: 6) for additional details

Appendix A and B

Appendix A

$$I_1 = \iint (4L_I-1)^2\,dA = \iint (16L_I^2 - 8L_I + 1)\,dA = (2A)\left[16\frac{2!}{4!} - \frac{8}{3!} + \frac{1}{2!}\right]$$

$$= (2A)\left[\frac{4}{3} - \frac{4}{3} + \frac{1}{2}\right] = A$$

$$\boxed{I_1 = \iint (4L_I-1)^2\,dA = \iint (4L_J-1)^2\,dA = \iint (4L_K-1)^2\,dA = A}$$

$$I_2 = \iint L_I(4L_J-1)\,dA = (2A)\left(\frac{4}{4!} - \frac{1}{3!}\right) = 0$$

$$\boxed{I_2 = \iint L_I(4L_J-1)\,dA = \iint L_J(4L_K-1)\,dA = \iint L_K(4L_I-1)\,dA = 0}$$

$$I_3 = \iint L_I(4L_I-1)\,dA = (2A)\left[4\frac{2!}{4!} - \frac{1}{3!}\right] = \frac{2A}{6} = \frac{A}{3}$$

$$\boxed{I_3 = \iint L_I(4L_I-1)\,dA = \iint L_J(4L_J-1)\,dA = \iint L_K(4L_K-1)\,dA = \frac{A}{3}}$$

$$I_4 = \iint (4L_I-1)(4L_J-1)\,dA = \iint (16L_IL_J - 4L_I - 4L_J + 1)\,dA$$

$$= (2A)\left[\frac{16}{4!} - \frac{4}{3!} - \frac{4}{3!} + \frac{1}{2!}\right] = \frac{2A}{6}\left[4 - 4 - 4 + 3\right] = -\frac{A}{3}$$

$$\boxed{I_4 = \iint (4L_I-1)(4L_J-1)\,dA = \iint (4L_J-1)(4L_K-1)\,dA = \iint (4L_K-1)(4L_I-1)\,dA = -A/3}$$

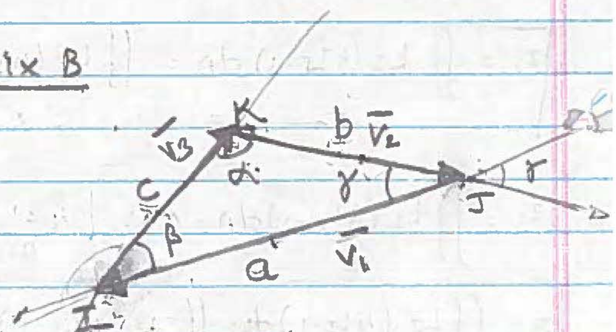
$$I_5 = \iint L_J^2\, dA = 2A \frac{2!}{4!} = \frac{A}{6}$$

$$\boxed{I_5 = \iint L_I^2\, dA + \iint L_J^2\, dA = \iint L_K^2\, dA = A/6}$$

$$I_6 = \iint L_I L_J\, dA = (2A)\frac{1!1!}{4!} = \frac{A}{12}$$

$$\boxed{I_6 = \iint L_I L_J\, dA = \iint L_J L_K\, dA = \iint L_K L_I\, dA = \frac{A}{12}}$$

Appendix B



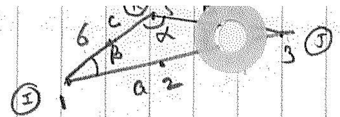
$$\bar{V}_1 = x_{IJ}\,\bar{i} + y_{IJ}\,\bar{j}$$
$$\bar{V}_2 = x_{JK}\,\bar{i} + y_{JK}\,\bar{j}$$
$$\bar{V}_3 = x_{KI}\,\bar{i} + y_{KI}\,\bar{j}$$

$$\bar{V}_1 \cdot \bar{V}_2 = (x_{IJ}x_{JK} + y_{IJ}y_{JK}) = ab\cos(180-\gamma) = -ab\cos\gamma$$
$$\bar{V}_2 \cdot \bar{V}_3 = (x_{JK}x_{KI} + y_{JK}y_{KI}) = bc\cos(180-\alpha) = -bc\cos\alpha$$
$$\bar{V}_1 \cdot \bar{V}_3 = (x_{IJ}x_{KI} + y_{IJ}y_{KI}) = ac\cos(180-\beta) = -ac\cos\beta$$

also:
$$a^2+b^2-c^2 = 2ab\cos\gamma \qquad x_{IJ}^2 + y_{IJ}^2 = a^2$$
$$b^2+c^2-a^2 = 2bc\cos\alpha \qquad x_{JK}^2 + y_{JK}^2 = b^2$$
$$a^2+c^2-b^2 = 2ac\cos\beta. \qquad x_{IK}^2 + y_{IK}^2 = c^2.$$

APPENDIX C: Stiffness matrix and right hand side vector for quadratic triangular element.

Quadratic Triangular Element



$$[K] = \frac{1}{12A}\begin{bmatrix} 3b^2 & -4bc\cos\alpha & bc\cos\alpha & 0 & ab\cos\gamma & -4ab\cos\gamma \\ & 4(a^2+b^2+c^2) & -4bc\cos\alpha & -8ab\cos\gamma & 0 & -8ac\cos\beta \\ & & 3c^2 & -4ac\cos\beta & ac\cos\beta & 0 \\ & & & 4(a^2+b^2+c^2) & -4ac\cos\beta & -8bc\cos\alpha \\ & & & & 3a^2 & -4ab\cos\gamma \\ & & & & & 4(a^2+b^2+c^2) \end{bmatrix}$$

$$Q_i = \frac{1}{6}\begin{Bmatrix} Q_a + Q_c \\ 4Q_a \\ Q_a + Q_b \\ 4Q_b \\ Q_b + Q_c \\ 4Q_c \end{Bmatrix} \qquad f_i = \frac{Af^e}{3}\begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{Bmatrix}$$

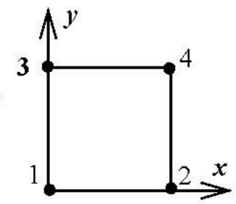
$Q_a = (q_n^a a)$ = Total heat flow out al side a

$\{R_i^{(e)}\} = \{Q_i\} + \{f_i\}$

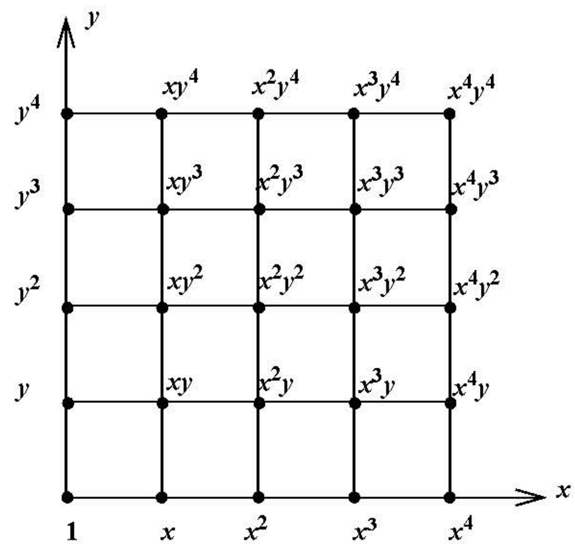
Rectangular Element For Second Order System in 2-D

Linear:

$$u(x) = C_0 + C_1x + C_2y + C_3xy$$

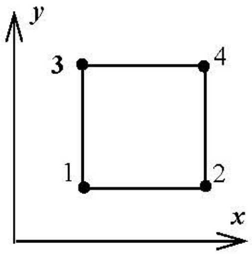


Higher order polynomials

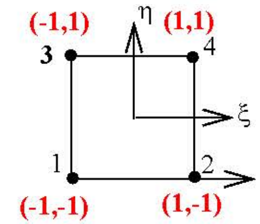


Natural Coordinate

Linear transformation

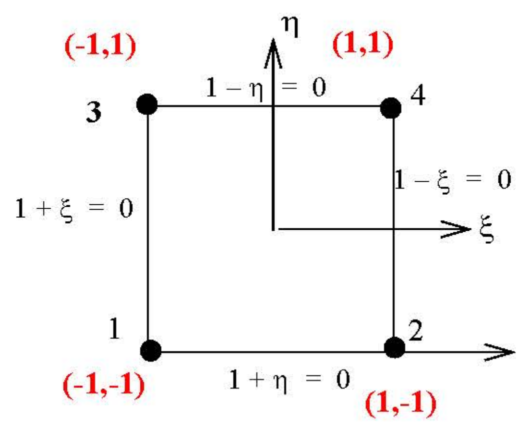


$$\xi = \frac{2x - (x_2^{(e)} + x_1^{(e)})}{(x_2^{(e)} - x_1^{(e)})} \qquad \eta = \frac{2y - (y_3^{(e)} + y_1^{(e)})}{(y_3^{(e)} - y_1^{(e)})}$$



Approximation of primary variables

Linear



$$\psi_1 = \frac{(1-\xi)(1-\eta)}{(1-(-1))(1-(-1))} = \frac{1}{4}(1-\xi)(1-\eta)$$

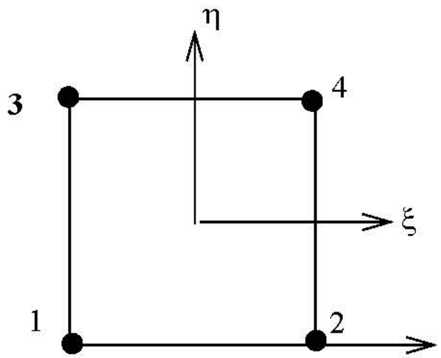
$$\psi_2 = \frac{(1+\xi)(1-\eta)}{(1+1)(1-(-1))} = \frac{1}{4}(1+\xi)(1-\eta)$$

$$\psi_3 = \frac{(1-\xi)(1+\eta)}{(1-(-1))(1+1)} = \frac{1}{4}(1-\xi)(1+\eta)$$

$$\psi_4 = \frac{(1+\xi)(1+\eta)}{(1+1)(1+1)} = \frac{1}{4}(1+\xi)(1+\eta)$$

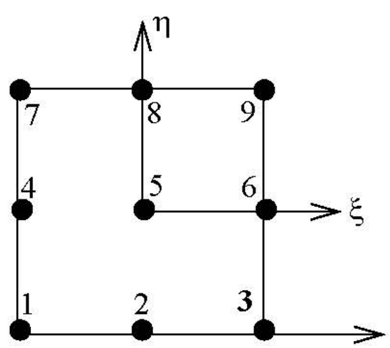
Tensor Product

Linear approximation of primary variable.



$$\begin{bmatrix}\psi_1 & \psi_3\\ \psi_2 & \psi_4\end{bmatrix} = \begin{bmatrix}\frac{1-\xi}{2}\\ \frac{1+\xi}{2}\end{bmatrix}\begin{bmatrix}\frac{1-\eta}{2} & \frac{1+\eta}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{4}(1-\xi)(1-\eta) & \frac{1}{4}(1-\xi)(1+\eta)\\ \frac{1}{4}(1+\xi)(1-\eta) & \frac{1}{4}(1+\xi)(1+\eta)\end{bmatrix}$$

Quadratic approximation of primary variable



$$\begin{bmatrix}\psi_1 & \psi_4 & \psi_7\\ \psi_2 & \psi_5 & \psi_8\\ \psi_3 & \psi_6 & \psi_9\end{bmatrix} = \begin{bmatrix}-\left(\frac{\xi}{2}\right)(1-\xi)\\ (1-\xi^2)\\ \left(\frac{\xi}{2}\right)(1+\xi)\end{bmatrix}\begin{bmatrix}-\left(\frac{\eta}{2}\right)(1-\eta) & (1-\eta^2) & \left(\frac{\eta}{2}\right)(1+\eta)\end{bmatrix}$$

Isoparametric Formulation (Reddy's Section 9.3)

Coordinate Transformation:

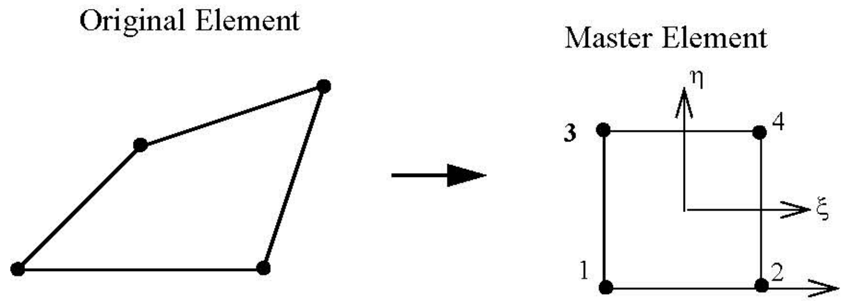
$$x = \sum_{i=1}^{N} x_i^{(e)}\phi_i(\xi,\eta) \qquad y = \sum_{i=1}^{N} y_i^{(e)}\phi_i(\xi,\eta)$$

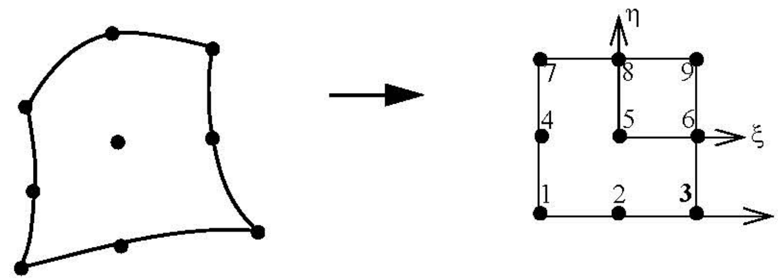
Primary variable approximation:

$$u = \sum_{i=1}^{M} u_i^{(e)}\psi_i(\xi,\eta)$$

Isoparametric:

$M = N$ and $\phi_i(\xi,\eta) = \psi_i(\xi,\eta)$





$$dx = \left(\sum_{i=1}^{N} x_i^{(e)}\frac{\partial\phi_i}{\partial\xi}\right)d\xi + \left(\sum_{i=1}^{N} x_i^{(e)}\frac{\partial\phi_i}{\partial\eta}\right)d\eta$$

$$dy = \left(\sum_{i=1}^{N} y_i^{(e)}\frac{\partial\phi_i}{\partial\xi}\right)d\xi + \left(\sum_{i=1}^{N} y_i^{(e)}\frac{\partial\phi_i}{\partial\eta}\right)d\eta$$

$$\frac{\partial\psi_i}{\partial\xi} = \frac{\partial\psi_i}{\partial x}\frac{\partial x}{\partial\xi} + \frac{\partial\psi_i}{\partial y}\frac{\partial y}{\partial\xi} \qquad \frac{\partial\psi_i}{\partial\eta} = \frac{\partial\psi_i}{\partial x}\frac{\partial x}{\partial\eta} + \frac{\partial\psi_i}{\partial y}\frac{\partial y}{\partial\eta}$$

or

$$\begin{Bmatrix}\frac{\partial\psi_i}{\partial\xi}\\ \frac{\partial\psi_i}{\partial\eta}\end{Bmatrix} = \begin{bmatrix}\frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi}\\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta}\end{bmatrix}\begin{Bmatrix}\frac{\partial\psi_i}{\partial x}\\ \frac{\partial\psi_i}{\partial y}\end{Bmatrix} = \begin{bmatrix}\left(\sum_{i=1}^{N} x_i^{(e)}\frac{\partial\phi_i}{\partial\xi}\right) & \left(\sum_{i=1}^{N} y_i^{(e)}\frac{\partial\phi_i}{\partial\xi}\right)\\ \left(\sum_{i=1}^{N} x_i^{(e)}\frac{\partial\phi_i}{\partial\eta}\right) & \left(\sum_{i=1}^{N} y_i^{(e)}\frac{\partial\phi_i}{\partial\eta}\right)\end{bmatrix}\begin{Bmatrix}\frac{\partial\psi_i}{\partial x}\\ \frac{\partial\psi_i}{\partial y}\end{Bmatrix} = [J]\begin{Bmatrix}\frac{\partial\psi_i}{\partial x}\\ \frac{\partial\psi_i}{\partial y}\end{Bmatrix}$$

$|J|$ = Jacobian = Determinant of [J] $= \left(\frac{\partial x}{\partial\xi}\right)\left(\frac{\partial y}{\partial\eta}\right) - \left(\frac{\partial y}{\partial\xi}\right)\left(\frac{\partial x}{\partial\eta}\right) > 0$ ---See Example 9.3.1

$$\begin{Bmatrix}\frac{\partial\psi_i}{\partial x}\\ \frac{\partial\psi_i}{\partial y}\end{Bmatrix} = [J]^{-1}\begin{Bmatrix}\frac{\partial\psi_i}{\partial\xi}\\ \frac{\partial\psi_i}{\partial\eta}\end{Bmatrix} \qquad dxdy = |J|\,d\xi d\eta$$

Poisson's Equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f$$

$$B^{(e)}(\mathrm{v}, u) = \iint_{A^{(e)}} \left(\frac{\partial \mathrm{v}}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y}\frac{\partial u}{\partial y} \right) dx\,dy \qquad l^{(e)}(\mathrm{v}) = \iint_{A^{(e)}} f\mathrm{v}\,dx\,dy + \oint q_n \mathrm{v}\,ds$$

where q_n is the specified value of $\frac{\partial u}{\partial n}$

Evaluation of stiffness matrix and right hand side vector

$$K_{ij}^{(e)} = B^{(e)}(\psi_i, \psi_j) = \iint_{A^{(e)}} \left(\frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} + \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} \right) dx\,dy = \iint_{A^{(e)}} \begin{Bmatrix} \frac{\partial \psi_i}{\partial x} \\ \frac{\partial \psi_i}{\partial y} \end{Bmatrix}^T \begin{Bmatrix} \frac{\partial \psi_j}{\partial x} \\ \frac{\partial \psi_j}{\partial y} \end{Bmatrix} dx\,dy \qquad R_i^{(e)} = l^{(e)}(\psi_i) = \iint_{A^{(e)}} f\psi_i\,dx\,dy + \oint q_n \psi_i\,ds$$

Substituting $\begin{Bmatrix} \frac{\partial \psi_i}{\partial x} \\ \frac{\partial \psi_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial \psi_i}{\partial \xi} \\ \frac{\partial \psi_i}{\partial \eta} \end{Bmatrix}$ $\quad dx\,dy = |J|\,d\xi\,d\eta$ and assuming $f = f^{(e)}$, the value at the center of the element.

$$K_{ij}^{(e)} = \int_{-1}^{1}\int_{-1}^{1} \begin{Bmatrix} \frac{\partial \psi_i}{\partial \xi} \\ \frac{\partial \psi_i}{\partial \eta} \end{Bmatrix}^T [[J]^{-1}]^T [J]^{-1} \begin{Bmatrix} \frac{\partial \psi_j}{\partial \xi} \\ \frac{\partial \psi_j}{\partial \eta} \end{Bmatrix} |J|\,d\xi\,d\eta = \int_{-1}^{1}\int_{-1}^{1} F_{ij}(\xi, \eta)\,d\xi\,d\eta \qquad R_i^{(e)} = f^{(e)} \int_{-1}^{1}\int_{-1}^{1} \psi_i |J|\,d\xi\,d\eta + \oint q_n \psi_i\,ds$$

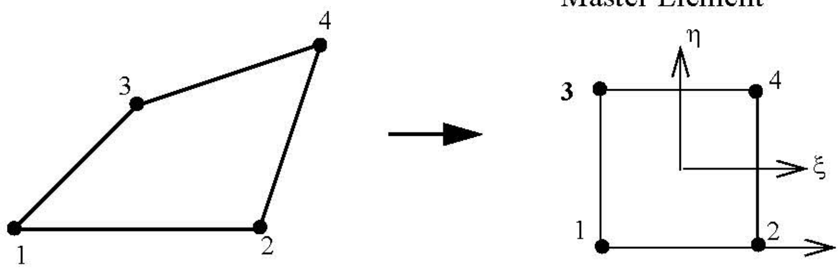
Analytical evaluation of stiffness matrix and right hand side vector

Assumption: The original element is a quadrilateral hence we can use linear coordinate transformation.

$$x = \sum_{i=1}^{N} x_i^{(e)}\phi_i(\xi,\eta) \qquad y = \sum_{i=1}^{N} y_i^{(e)}\phi_i(\xi,\eta)$$

Original Element

Master Element



$$\phi_1 = \frac{(1-\xi)(1-\eta)}{(1-(-1))(1-(-1))} = \frac{1}{4}(1-\xi)(1-\eta)$$

$$\phi_2 = \frac{(1+\xi)(1-\eta)}{(1+1)(1-(-1))} = \frac{1}{4}(1+\xi)(1-\eta)$$

$$\phi_3 = \frac{(1-\xi)(1+\eta)}{(1-(-1))(1+1)} = \frac{1}{4}(1-\xi)(1-\eta)$$

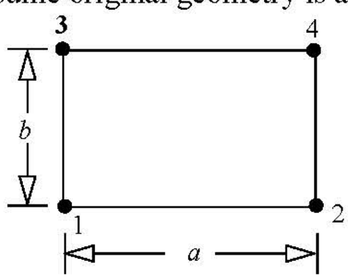
$$\phi_4 = \frac{(1+\xi)(1+\eta)}{(1+1)(1+1)} = \frac{1}{4}(1+\xi)(1+\eta)$$

$$J_{11} = \sum_{i=1}^{N} x_i^{(e)}\frac{\partial\phi_i}{\partial\xi} = \frac{x_{12}(1-\eta)+x_{34}(1+\eta)}{4} \qquad J_{12} = \sum_{i=1}^{N} y_i^{(e)}\frac{\partial\phi_i}{\partial\xi} = \frac{y_{12}(1-\eta)+y_{34}(1+\eta)}{4}$$

$$J_{21} = \sum_{i=1}^{N} x_i^{(e)}\frac{\partial\phi_i}{\partial\eta} = \frac{x_{13}(1-\xi)+x_{24}(1+\xi)}{4} \qquad J_{22} = \sum_{i=1}^{N} y_i^{(e)}\frac{\partial\phi_i}{\partial\eta} = \frac{y_{13}(1-\xi)+y_{24}(1+\xi)}{4}$$

$$x_{ij} = x_j - x_i \qquad y_{ij} = y_j - y_i$$

Assume original geometry is a rectangle

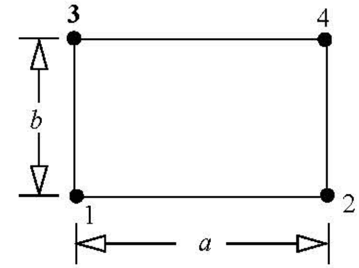


$$x_{12} = x_{34} = \mathrm{a}$$
$$x_{13} = x_{24} = 0$$
$$y_{12} = y_{34} = 0$$
$$y_{13} = y_{24} = \mathrm{b}$$

$$[J] = \begin{bmatrix} a/2 & 0 \\ 0 & b/2 \end{bmatrix} \qquad [J]^{-1} = \begin{bmatrix} 2/a & 0 \\ 0 & 2/b \end{bmatrix} \qquad |J| = \frac{ab}{4}$$

$$K_{ij}^{(e)} = \frac{b}{a}\int_{-1}^{1}\int_{-1}^{1}\left[\frac{\partial\psi_i}{\partial\xi}\frac{\partial\psi_j}{\partial\xi}\right]d\xi\, d\eta + \frac{a}{b}\int_{-1}^{1}\int_{-1}^{1}\left[\frac{\partial\psi_i}{\partial\eta}\frac{\partial\psi_j}{\partial\eta}\right]d\xi\, d\eta \qquad R_i^{(e)} = \frac{f^{(e)}ab}{4}\int_{-1}^{1}\int_{-1}^{1}\psi_i\, d\xi\, d\eta + \oint q_n\psi_i\, ds$$

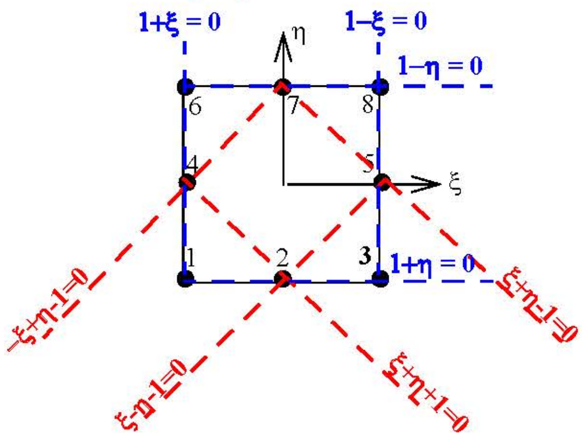
C.7 Obtain $K_{11}^{(e)}$, $K_{12}^{(e)}$, and $R_1^{(e)}$ for a linear rectangular element shown for use in solution of Poisson's equation.



Serendipity Elements (Section 9.2.3 of Reddy)

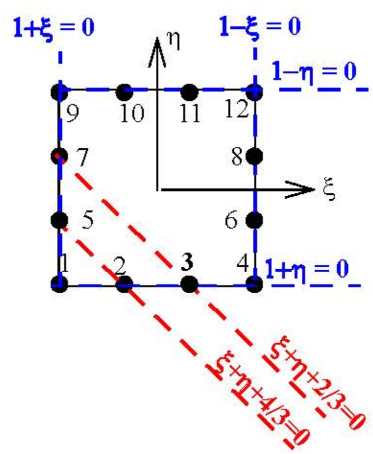
- The internal nodes increase the degree of freedoms (number of unknowns) with minimal impact on accuracy. The objective is to obtain the same order of polynomial without the internal nodes.
- Cannot use tensor product to get interpolation functions.

Quadratic serendipity element



$$\psi_1 = \frac{(1-\xi)(1-\eta)(\xi+\eta+1)}{(1+1)(1+1)(-1-1+1)} = -\left(\frac{1}{4}\right)(1-\xi)(1-\eta)(\xi+\eta+1)$$

$$\psi_2 = \frac{(1-\xi)(1-\eta)(1+\xi)}{(1-0)(1+1)(1+0)} = \left(\frac{1}{2}\right)(1-\xi^2)(1-\eta)$$

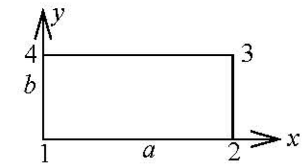


$$\psi_1 = \frac{(1-\xi)(1-\eta)[9(\xi^2+\eta^2)-10]}{(2)(2)(8)} = \frac{1}{32}(1-\xi)(1-\eta)[9(\xi^2+\eta^2)-10]$$

Stationary value of a functional with second order derivatives in rectangular coordinates

Green's Formula

$$\iint_A f\frac{\partial g}{\partial x}dx\,dy = \oint_\Gamma fg\,dy - \iint_A g\frac{\partial f}{\partial x}dx\,dy \qquad \iint_A f\frac{\partial g}{\partial y}dx\,dy = -\oint_\Gamma fg\,dx - \iint_A g\frac{\partial f}{\partial y}dx\,dy$$



$$I(u) = \iint_A H(u_{,xx}, u_{,xy}, u_{,yy}, u_{,x}, u_{,y}, u, x, y)\,dx\,dy \qquad \text{where} \qquad u_{,xx} = \partial^2 u/\partial x^2 \qquad u_{,yy} = \partial^2 u/\partial y^2 \qquad u_{,xy} = \partial^2 u/\partial x\partial y$$

Differential Equation

$$\boxed{\frac{\partial^2}{\partial x^2}\left(\frac{\partial H}{\partial u_{,xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial H}{\partial u_{,yy}}\right) + \frac{\partial^2}{\partial x\partial y}\left(\frac{\partial H}{\partial u_{,xy}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u_{,x}}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial u_{,y}}\right) + \left(\frac{\partial H}{\partial u}\right) = 0}$$

Boundary Conditions

$$\frac{\partial H}{\partial u_{,xx}} = 0 \quad \text{or} \quad \delta u_{,x} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u_{,xx}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial u_{,xy}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u_{,x}}\right) = 0 \quad \text{or} \quad \delta u = 0$$

at $x = 0$ and $x = a$

$$\frac{\partial H}{\partial u_{,yy}} = 0 \quad \text{or} \quad \delta u_{,y} = 0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial H}{\partial u_{,yy}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u_{,xy}}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial u_{,y}}\right) = 0 \quad \text{or} \quad \delta u = 0$$

at $y = 0$ and $y = b$

Corner Conditions

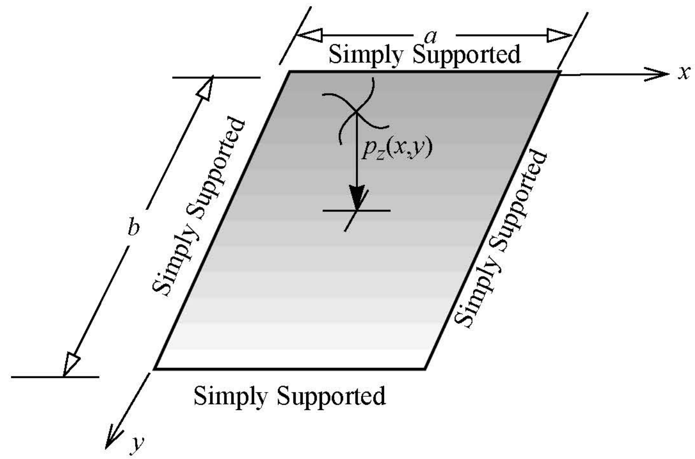
$$\boxed{\left(\frac{\partial H}{\partial u_{,xy}}\right)_k = 0 \quad \text{or} \quad \delta u_k = 0 \qquad k = 1 \text{ to } 4}$$

C.8 Obtain the boundary value problem for thin plate bending starting with the functional given below.

$$\Omega(w) = \frac{D}{2}\iint_A \left\{ \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right\} dx\,dy - \iint_A p_z w\,dx\,dy$$

where, w is the plate deflection in the z direction; p_z is the distribute force per unit area on the plate; $D = Eh^3/[12(1-\nu^2)]$ is the plate bending rigidity; h is the thickness of the plate; E is the modulus of elasticity; and ν is the Poisson's ratio.

Boundary value problem for simply supported plates



The boundary value problem for the plate shown can be written as

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_z(x,y)}{D}$$

On $x = 0$ $\quad w(0,y) = 0 \quad \dfrac{\partial^2 w}{\partial x^2}(0,y) = 0$

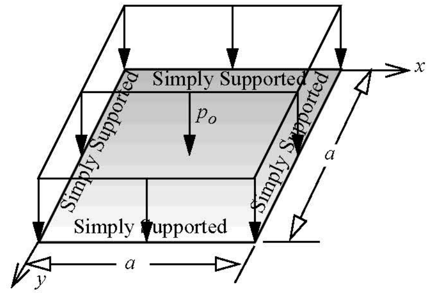
On $x = a$ $\quad w(a,y) = 0 \quad \dfrac{\partial^2 w}{\partial x^2}(a,y) = 0$

On $y = 0$ $\quad w(x,0) = 0 \quad \dfrac{\partial^2 w(x,0)}{\partial y^2} = 0$

On $y = b$ $\quad w(x,b) = 0 \quad \dfrac{\partial^2 w(x,b)}{\partial y^2} = 0$

C.9 A square plate is uniformly loaded and is simply supported on all sides as shown. Using Rayleigh-Ritz method and the given approximation determine the deflection w at the center of the plate and the potential energy for each case.

$$w(x, y) = C_1 \sin\pi(x/a)\sin\pi(y/a) + C_2 \sin\pi(3x/a)\sin\pi(y/a) + C_3 \sin\pi(x/a)\sin\pi(3y/a)$$

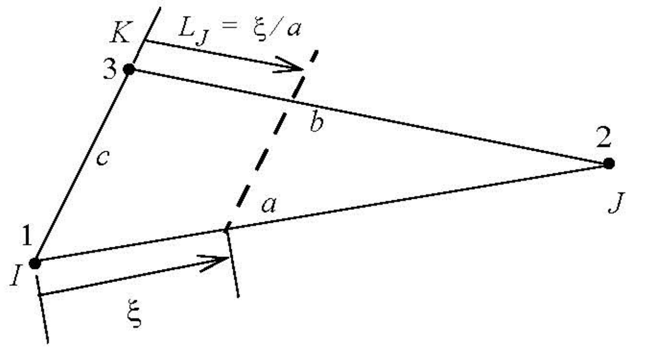


Conforming and non-conforming elements

- When the continuity of primary variables are ensured at the nodes and the boundary then the element is called a **conforming element**.
- When continuity of primary variables are only ensured at the nodes but not ensured across the boundary then the element is called **non-conforming element.**

Continuity of primary variable on boundary

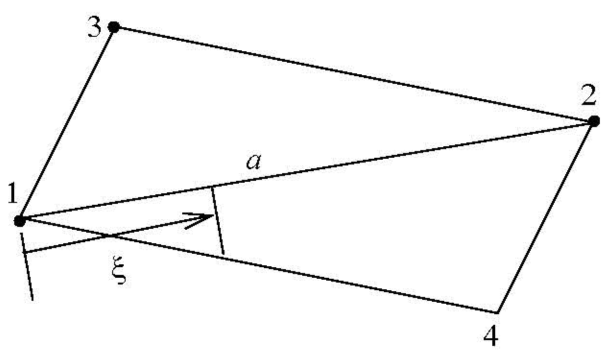
Linear Element



$$^{(e)} = \sum_{i=1}^{M} u_i^{(e)} \psi_i(L_I, L_J, L_K$$

Linear Element: $_1 = L_I \qquad \psi_2 = L_J \qquad \psi_3 = L$

On line 1-2: $^{(e)} = \left(1 - \frac{\xi}{a}\right) u_1^{(e)} + \frac{\xi}{a} u_2^{(e}$



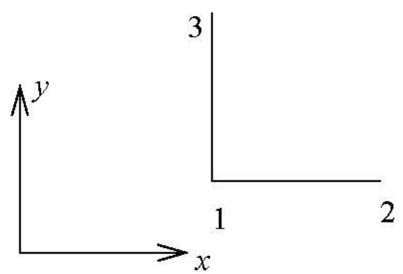
$$u^{(1)} = \left(1 - \frac{\xi}{a}\right) u_1 + \frac{\xi}{a} u_2$$
$$u^{(2)} = \left(1 - \frac{\xi}{a}\right) u_1 + \frac{\xi}{a} u_2$$

- The polynomial approximation will result in a continuity if u depends only on the nodes that are on the boundary line.
- For second order systems only continuity of u is needed but for 4th order systems u and its derivatives must be continuous.

Challenges for fourth order system in 2-D

Primary variables: u, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

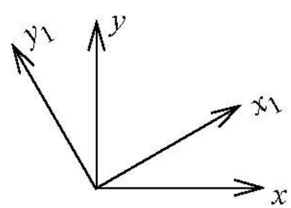
Secondary variables will have second order derivatives: $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial x \partial y}$, and third order derivatives. Secondary variables can be discontinuous.



Along line 1-2 $\frac{\partial u}{\partial y}$ will depend on nodal values of node 1 and 2. Thus $\frac{\partial^2 u}{\partial x \partial y}$ will depend upon nodes 1 and 2.

Along line 1-3 $\frac{\partial u}{\partial x}$ will depend on nodal values of node 1 and 3. Thus $\frac{\partial^2 u}{\partial y \partial x}$ will depend upon nodes 1 and 3.

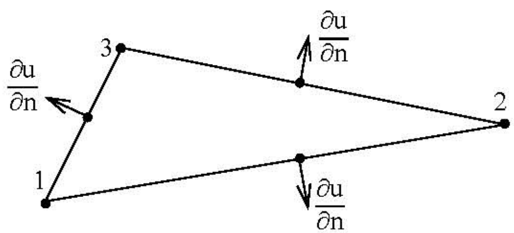
At node 1 we may not satisfy the condition $\frac{\partial^2 u}{\partial x \partial y}$ is equal to $\frac{\partial^2 u}{\partial y \partial x}$ because of independent values of node 2 and 3.



- Continuity of cross derivatives in different oriented coordinate system will require continuity of all second order derivatives.
- Continuity of all second order derivatives may imply that secondary variables must be continuous, while actual situation requires the secondary variable to be discontinuous.

Conforming elements for fourth order systems

Triangular element



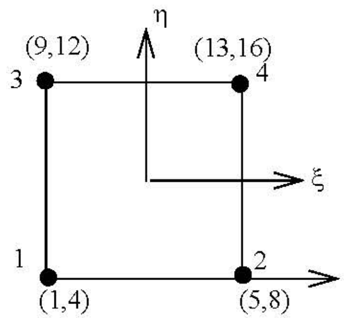
Each node has the following 6 degrees of freedom.

$$;\ \frac{\partial u}{\partial x};\ \frac{\partial u}{\partial y};\ \frac{\partial^2 u}{\partial x^2};\ \frac{\partial^2 u}{\partial y^2};\ \frac{\partial^2 u}{\partial x \partial y}$$

Each mid point has 1 degree of freedom of the normal derivative $\frac{\partial u}{\partial n}$.

Total degrees of freedom for the element = 21

Rectangular element



Each node has the following 4 degrees of freedom.

$$u;\ \frac{\partial u}{\partial \xi};\ \frac{\partial u}{\partial \eta};\ \frac{\partial^2 u}{\partial \xi \partial \eta}$$

Total degree of freedom = 16.

Can be used only for original element oriented as the master element shown.

Hermite Polynomials: $= \frac{1}{4}(\xi^3 - 3\xi + 2)$ $\quad H_1^{\{1\}} = \frac{1}{4}(\xi^3 - \xi^2 - \xi + 1)$ $\quad H_2^{\{0\}} = -\left(\frac{1}{4}\right)(\xi^3 - 3\xi - 2)$ $\quad H_2^{\{1\}} = \frac{1}{4}(\xi^3 + \xi^2 - \xi$

Tensor Product: $\begin{bmatrix} 1 & \psi_5 & \psi_9 & \psi_{13} \\ 2 & \psi_6 & \psi_{10} & \psi_{14} \\ 3 & \psi_7 & \psi_{11} & \psi_{15} \\ 4 & \psi_8 & \psi_{12} & \psi_{16} \end{bmatrix} = \begin{vmatrix} H_1^{\{0\}}(\xi) \\ H_1^{\{1\}}(\xi) \\ H_2^{\{0\}}(\xi) \\ H_2^{\{1\}}(\xi) \end{vmatrix} \left[H_1^{\{0\}}(\eta)\ H_1^{\{1\}}(\eta)\ H_2^{\{0\}}(\eta)\ H_2^{\{0\}}(\right.$

Possible reasons for using non-conforming elements

1. Imposing complete compatibility may result in a situation where discontinuity in secondary variable cannot be simulated.
2. Convergence rate is too slow. Alternatively the number of degree of freedom are too many.
3. Different order of differential equations govern the physics of the problem, thus at the junction there can be a discontinuity in the derivative of the function.
4. The dimension in one direction is small compared to the other direction, thus do not want to commit lots of resources in the direction of the smaller dimension.

Some observations related to non-conforming elements

- When non-conforming elements are used convergence is not guaranteed. That is, mesh refinement may not increase accuracy.
- Patch test is sometime used to check convergence issues for non-conforming elements. Patch test is a numerical test of a small assembly of element subjected to known analytical solution and the numerical results are checked for convergence to analytical solution.
- Can the higher order differential equation be replaced by a set of lower order differential equation. For example, the classical beam bending differential equation is a fourth order differential equation. It can be replaced by two second order differential equation in Timoshenko beams.