Time Dependent Problems

There are two approaches we can take for time dependent problems: (see Reddy' Section 6-2)

1. Treat time like any other space variable and obtain the weak form by integrating over time and space.

$$u = \sum_{i=1}^{n} u_i^{(e)} \phi_i(x, y, z, t) \tag{1}$$

2. Assume that time and space can be separated. Weak form is constructed by integrating over space and we obtain ordinary differential equations in time that are solved by a variety of approaches that we will study.

$$u = \sum_{i=1}^{n} u_i^{(e)}(t) \phi_i(x, y, z) \tag{2}$$

Approach one is applicable to all types of time dependent problems. The primary disadvantage of approach one is the rapid growth of degrees of freedom in an element as seen in Table 1. There are however papers that can be found in the literature on this approach.

Table 1: Degrees of freedom per element

	1-D	2-D	3-D	4-D
Linear Element	2^1=2	2^2=4	2^3=8	2^4=16
Quadratic Element	3^1=3	3^2=9	3^3=27	3^4=81

The difficulty with approach two is that all problems are not amenable to separation of variables as assumed, specifically: hyperbolic equations. For example, the wave propagation through solids in 1-D has a solution of the type $f(x-ct)$ or $g(x+ct)$, where c is the velocity of sound through the solid. Other types of problems for which approach 1 would be better than approach 2 could be transient response problems (parabolic equations).

However, approach two is the predominantly approach used in FEM. If time and space are separable than the two approaches produce the same results, but approach 2 uses less computer resources. There are creative ways of using approach two to overcome the difficulty when time and space are not separable.

C.1 The differential equation below has many applications

$$m\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) - f(x) = 0 \qquad (3)$$

where, m, c, and a can be functions of x. f and $c = 0$ ---vibration of an elastic string; m = mass per unit length and $a = EA$; m=0----heat flow in a bar. Using the approximation $u = \sum_{i=1}^{n} u_i^{(e)}(t)\phi_i(x)$ obtain the ordinary differential equations in matrix form shown below.

$$[M]\left\{\frac{d^2 u}{dt^2}\right\} + [C]\left\{\frac{du}{dt}\right\} + [K]\{u\} = \{R\} \qquad (4)$$

where,

$$M_{ji}^{(e)} = \int_{x_1^{(e)}}^{x_2^{(e)}} m\phi_i\phi_j dx \qquad C_{ji}^{(e)} = \int_{x_1^{(e)}}^{x_2^{(e)}} c\phi_i\phi_j dx \qquad K_{ji}^{(e)} = \int_{x_1^{(e)}}^{x_2^{(e)}} m\phi_i\phi_j dx \qquad R_j^{(e)} = \int_{x_1^{(e)}}^{x_2^{(e)}} f(x)\phi_j dx + \phi_j(x_2^{(e)})Q_2^{(e)} + \phi_j(x_1^{(e)})Q_1^{(e)} \qquad (5)$$

$$Q_1^{(e)} = -\left[a\frac{\partial u}{\partial x}\right]\Bigg|_{x_1^{(e)}} \qquad Q_2^{(e)} = \left[a\frac{\partial u}{\partial x}\right]\Bigg|_{x_2^{(e)}} \qquad (6)$$

The matrices $[M]$ and $[K]$ are symmetric, positive definite matrices. The matrix $[C]$ may not be positive definite.

Classification of problems

Reference: O.C. Zienkiewicz: “The finite element method” 3rd edition McGraw-Hill Company. New York.

The set of problems represented Equation (4) by can be classified as:

1. Free response: {R}={0}--eigenvalue problem
2. Periodic response: {R(t)} is periodic
3. Transient response: {R(t)} is arbitrary.

The solution for free response and periodic response is very similar to the approach we take to solve differential equations.

Numerical Methods

The starting point for finding eigenvalues numerically requires the problem in following matrix form:

$$[H]\{x\} = \lambda\{x\} \tag{7}$$

where, λ is an eigenvalue and $\{x\}$ is the associated eigenvector.

1. Several of the methods discussed below are designed to get the matrix equations into the form given by Equation (7).
2. Another major idea that influences the solution is that we would like to get the matrix $[H]$ as a symmetric matrix.

I. Eigenvalues

Eigenvalues-Free vibrations: [C]=0 and {R}=0

Equation (4) can be written as:

$$[M]\left\{\frac{d^2u}{dt^2}\right\} + [K]\{u\} = 0 \tag{8}$$

We seek a solution in the following form:

$$\{u\} = \{x\}e^{\bar{i}\omega t} \qquad \text{where} \qquad \bar{i} = \sqrt{-1} \tag{9}$$

Substituting Equation (4) into Equation (8) we obtain:

$$-\omega^2[M]\{x\} + [K]\{x\} = 0 \tag{10}$$

In structures ω represents the frequency of vibrations.

C.2 If $[K]$ and $[M]$ are symmetric matrices then show that the eigenvectors are orthogonal with respect to the two matrices.

Determination of eigenvalues

There are two approaches in casting Equation (10) into the form of Equation (7): we could invert [*K*] (used in consistent mass matrix) or we could invert [*M*] (used with lumped mass matrix).

Consistent mass matrix

The mass matrix in Equation (5) is called consistent mass matrix. We invert [*K*], use Cholesky's decomposition $[K] = [L][L]^T$ and Equation (4) can be written as

$$[H]\{y\} = \lambda\{y\} \qquad \text{where} \qquad [H] = [L]^{-1}[M][L]^{-T} \qquad \{x\} = [L]^{-T}\{y\} \qquad \lambda = \frac{1}{\omega^2}$$

The [*H*] matrix in the above equation is symmetric. We first find λ and $\{y\}$ and then obtain $\{x\}$

Lumped mass

The idea is to get a diagonal mass matrix for it is simple to invert. See Reddy Section 6.2.5 for two approaches: **Row-Sum lumping** and **proportional lumping**. We will only discuss the first.

The sum of the elements of each row of the consistent mass matrix (Equation (5)) is used as the diagonal term:

$$M_{ii}^{(e)} = \sum_j \int_{x_1^{(e)}}^{x_2^{(e)}} m\phi_i\phi_j dx = \int_{x_1^{(e)}}^{x_2^{(e)}} m\phi_i\left(\sum_j \phi_j\right) dx = \int_{x_1^{(e)}}^{x_2^{(e)}} m\phi_i dx \qquad (11)$$

where we used the property of Lagrange polynomial $\sum_j \phi_j = 1$. For a linear element, assuming constant mass per unit length, the mass matrix is:

$$[M^{(e)}] = \frac{mh^{(e)}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (12)$$

The physical significance of the above equation is that you take half the mass of the element to each node. In a similar manner for a linear triangular element you take 1/3 the mass of each element to the node and for a tetrahedron 1/4 the mass to each node. For quadratic and higher order polynomials the mass distribution at each node is somewhat different. (See Reddy Section 6.2.5). The global mass matrix will be a diagonal mass matrix. We invert the matrix and once more obtain:

$$[H]\{y\} = \lambda\{y\} \qquad \text{where} \qquad [H] = [M]^{-\frac{1}{2}}[K][M]^{-\frac{1}{2}} \qquad \{x\} = [M]^{-\frac{1}{2}}\{y\} \qquad \lambda = \omega^2$$

Once more [*H*] is a symmetric matrix. We first find λ and $\{y\}$ and then obtain $\{x\}$

Eigenvalues-First order equations (Heat conduction problem): {R}=0 and {M}=0

$$[C]\left\{\frac{du}{dt}\right\} + [K]\{u\} = 0 \tag{13}$$

The matrix [C] is symmetric but may not be positive definite. We seek a solution in the following form: $\{u\} = \{x\}e^{-\omega t}$ Using Cholesky's decomposition we once more obtain

$$[H]\{y\} = \lambda\{y\} \qquad \text{where} \qquad [H] = [L]^{-1}[C][L]^{-T} \qquad \{x\} = \{x\} = [L]^{-T}\{y\} \qquad \lambda = \frac{1}{\omega}$$

Free response of a damped systems: {R}=0

Equation (4) can be written as

$$[M]\left\{\frac{d^2u}{dt^2}\right\} + [C]\left\{\frac{du}{dt}\right\} + [K]\{u\} = 0 \tag{14}$$

We seek a solution in the following form:

$$\{u\} = \{x\}e^{\omega t} \tag{15}$$

We obtain:

$$(\omega^2[M] + \omega[C] + [K])\{x\} = 0$$

ω are the eigenvalues of the system, which in general will be complex. Let this be $\omega = \omega_1 + \bar{i}\,\omega_2$. Thus the solution of Equation (15) becomes

$$\{u\} = \{x\}e^{(\omega_1 + \bar{i}\,\omega_2)t} = \{x\}e^{\omega_1 t}(cos\omega_2 t + \bar{i}\, sin\omega_2 t) \tag{16}$$

If ω_1 is negative then the amplitude of the oscillation dies down with time, which is the case for most vibration problems. However, if ω_1 is positive than the amplitude of the oscillation keeps increasing with time—this is the phenomena of an aerodynamic instability called flutter.

In structures the damping coefficient *c* is hard to determine. So the following is done:

$$[C] = \alpha[M] + \beta[K] \tag{17}$$

where, α and β are experimentally determined constants. Damping represented by Equation (17) is called **Rayleigh damping**.

II Forced periodic response: (Harmonic forces)

The forcing function is periodic in nature as below.

$$\{R\} = \{f\}e^{\omega t} \tag{18}$$

where ω is complex and known.

Equation (4) can be written as

$$[M]\left\{\frac{d^2u}{dt^2}\right\} + [C]\left\{\frac{du}{dt}\right\} + [K]\{u\} = \{f\}e^{\omega t} \tag{19}$$

We seek a solution in the following form: $\{u\} = \{x\}e^{\omega t}$

We obtain:

$$(\omega^2[M] + \omega[C] + [K])\{x\} = [D]\{x\} = \{f\} \qquad or \qquad \{x\} = [D]^{-1}\{f\} \tag{20}$$

Note with ω, $[M]$, $[C]$, and $[K]$ known, the $[D]$ matrix is known and can be inverted. However, the matrix $[D]$ will be a complex matrix and thus complex arithmetic will have to be used. One could write $\omega = \omega_1 + \bar{i}\,\omega_2$ and write Equation (20) as two sets of real matrix equations.

III: Forced response indirect methods

There are two ways of approaching the forced response by indirect methods.

(i) Frequency response

(ii) Modal analysis

Frequency response

Any function can be represented by a Fourier series approximately if finite number of terms are used. We approximate the forcing function by the following Fourier series:

$$\{R\} = \sum_{j=1}^{n} \{f_j\}e^{\bar{i}\,\omega_j t} \tag{21}$$

Equation (4) can be written as

$$[M]\left\{\frac{d^2u}{dt^2}\right\} + [C]\left\{\frac{du}{dt}\right\} + [K]\{u\} = \sum_{j=1}^{n} \{f_j\}e^{\bar{i}\,\omega_j t} \tag{22}$$

We solved Equation (22) for 1 term in forced periodic response [see Equation (19)]. We do the same now for n terms.

If $n = \infty$ then the series in Equation (21) can be written as Fourier integral, also known as Fourier Transform. There are techniques designed for Fourier Transform available in the literature.

Modal Analysis

The response of a system is a linear combination of the eigenvectors of the system. The basic idea here is to find the coefficients of the linear combination. We set {R}=0 in Equation (4) and obtain the free response. This is solution of Equation (14) as given by Equation (15). For n eigenvalues we can write the total free response as:

$$\{u_{free}\} = \sum_{j=1}^{n} \{x_j\} e^{\omega_j t} \tag{23}$$

where, ω_j are the eigenvalues and $\{x_j\}$ are the corresponding eigenvectors. The solution of Equation (4) can be written as

$$\{u\} = \sum_{j=1}^{n} y_j(t)\{x_j\} \tag{24}$$

where, $y_j(t)$ are the undetermined coefficients of the linear combination that we seek to determine.
Substituting Equation (24) into Equation (4) we obtain:

$$\sum_{j=1}^{n} [M]\{x_j\}\frac{d^2 y_j}{dt^2} + \sum_{j=1}^{n} [C]\{x_j\}\frac{dy_j}{dt} + \sum_{j=1}^{n} [K]\{x_j\}y_j = \{R\} \tag{25}$$

We pre-multiply the above equation by $\{x_i\}^T$ to obtain

$$\sum_{j=1}^{n} \{x_i\}^T[M]\{x_j\}\frac{d^2 y_j}{dt^2} + \sum_{j=1}^{n} \{x_i\}^T[C]\{x_j\}\frac{dy_j}{dt} + \sum_{j=1}^{n} \{x_i\}^T[K]\{x_j\}y_j = \{x_i\}^T\{R\} \tag{26}$$

As the eigenvectors are orthogonal with regard to the matrices the following equations are valid:

$$\{x_i\}^T[M]\{x_j\} = 0 \qquad \{x_i\}^T[C]\{x_j\} = 0 \qquad \{x_i\}^T[K]\{x_j\} = 0 \qquad i \neq j \tag{27a}$$

We define the following for $i = j$:

$$\{x_i\}^T[M]\{x_i\} = m_i \qquad \{x_i\}^T[C]\{x_i\} = c_i \qquad \{x_i\}^T[K]\{x_i\} = k_i \qquad \{x_i\}^T\{R\} = r_i \tag{27b}$$

Substituting we obtain:

$$m_i\frac{d^2 y_i}{dt^2} + c_i\frac{dy_i}{dt} + k_i y_i = r_i \qquad i = 1 \text{ to n} \tag{28}$$

Equation (28) represents n ordinary differential equations that can be solved to obtain $y_i(t)$ and then from Equation (24) we obtain the solution $u(t)$ the response of the system.

III: Forced response direct methods.

In direct methods the time derivatives are written using finite differences. We can write the time derivatives at $t = t_n$ and $t = t_{n+1}$ as:

$$\left\{\frac{du}{dt}\right\}_n = \frac{\{u\}_{n+1} - \{u\}_n}{\Delta t_{n+1}} \qquad t_n \le t \le t_{n+1} \qquad \text{Forward Difference}$$

$$\left\{\frac{du}{dt}\right\}_{n+1} = \frac{\{u\}_{n+1} - \{u\}_n}{\Delta t_{n+1}} \qquad t_n \le t \le t_{n+1} \qquad \textit{Backward}\ \text{Difference}$$

$$\theta\left\{\frac{du}{dt}\right\} + (1-\theta)\left\{\frac{du}{dt}\right\}_{n+1} = \frac{\{u\}_{n+1} - \{u\}_n}{\Delta t_{n+1}} \qquad \theta = \begin{cases} 1 & \textit{Backward}\ \text{Difference} \\ 0 & \text{Forward Difference} \end{cases}$$

where $\Delta t_{n+1} = t_{n+1} - t_n$. The choice of Δt_{n+1} and θ are important in stability and accuracy of iterative solution schemes.