Review

Variational Methods in Engineering

Variational methods is the name of mathematics by which we find extreme values.

1. To find an extreme value we need to compare.

2. The comparison is in the immediate neighborhood and will give us only a local extremum value and not a global extremum.

Independent set of variables: When a variable in a set cannot be represented as a function of the other variables. $u_1, u_2, u_3,\ldots\ldots u_n$ are independent variables if there exist no non-zero values of a's for which the following is not true:

$$a_1 u_1 + a_2 u_2 + a_3 u_3 \bullet \quad \bullet \quad \bullet \quad \bullet \quad + a_n u_n = 0$$

Generalized Coordinates: Any set of independent variables (parameters) that describes the motion or geometry of the system.

Degrees of Freedom: Number of variables needed to describe the behavior of the system.

Kinematic Conditions: Any condition that limits the change in geometry.

Kinematically admissible function: A function that is continuous and satisfies all kinematic conditions.

- Actual movement: d-process: dx, ds , $du, \partial u \ldots\ldots$. Derivatives can be obtained from the functions and hence function and derivatives are not independent.

- **Virtual displacement** or the δ-process: Imaginary displacement in which function and its derivatives are independent. A virtual displacement must be kinematically admissible.

- **Stationary value of a function**: The first variation of the function is zero. The displacement must be reversible at the stationary point.

- **Extremum value of a function**: Two possibilities: (*i*) The first variation of the function is zero and the sign of second variation needs to be specified for determining maximum or minimum value. (*ii*) First variation is not zero and the displacement is not reversible-- happens on boundary for monotonic functions.

- **Functional**: A rule of correspondence such that for each function value $u(x)$ in R there is a unique value $F(u)$ in Ω.

To find the first variation of a functional [$\delta F(u)$]

1. During the process of variation the function u and its derivatives are independent.

2. After the variation, we are on a curve and u and its derivatives are not independent. We have to transfer the derivatives so that all terms in the domain integral are terms multiplying (δu).

1-D:
$$\int_a^b f\left(\frac{dg}{dx}\right)\,dx = (fg)\Big|_a^b - \int_a^b \left(\frac{df}{dx}\right)g\,dx$$

2-D:
$$\iint_A f\left(\frac{\partial g}{\partial x}\right)dxdy = \oint_\Gamma fg\,dy - \iint_A g\frac{\partial f}{\partial x}dxdy = \oint_\Gamma n_x(fg)ds - \iint_A \left(\frac{\partial f}{\partial x}\right)g\,dxdy$$

$$\iint_A f\left(\frac{\partial g}{\partial y}\right)dxdy = -\oint_\Gamma fg\,dx - \iint_A g\frac{\partial f}{\partial y}dxdy = \oint_\Gamma n_y(fg)ds - \iint_A \left(\frac{\partial f}{\partial y}\right)g\,dxdy$$

$$\left[\,n_x = 0;\ n_y = 1\,\right]$$
$$\left[\,ds = -dx\,\right]$$

$$\left[\,n_x = -1;\ n_y = 0\,\right]$$
$$\left[\,ds = dy\,\right]$$

$$\left[\,n_x = 1;\ n_y = 0\,\right]$$
$$\left[\,ds = dy\,\right]$$

$$\left[\,n_x = 0;\ n_y = -1\,\right]$$
$$\left[\,ds = dx\,\right]$$

3-D:
$$\iiint_T f\left(\frac{\partial g}{\partial x}\right)dxdydz = \iint_S n_x(fg)dS - \iiint_T \left(\frac{\partial f}{\partial x}\right)g\,dxdydz$$

$$\iiint_T f\left(\frac{\partial g}{\partial x}\right)dxdydz = \iint_S n_x(fg)dS - \iiint_T \left(\frac{\partial f}{\partial x}\right)g\,dxdydz$$

$$\iiint_T f\left(\frac{\partial g}{\partial z}\right)dxdydz = \iint_S n_z(fg)dS - \iiint_T \left(\frac{\partial f}{\partial z}\right)g\,dxdydz$$

3. Virtual change (δu) must satisfy all kinematic constraints.

4. For definite integrals variation (δ) and integral are commutable.

5. Variation (δ) and differentiation are commutable.

6. All boundary terms in the first variation of a functional (integral) are product of secondary variable and the
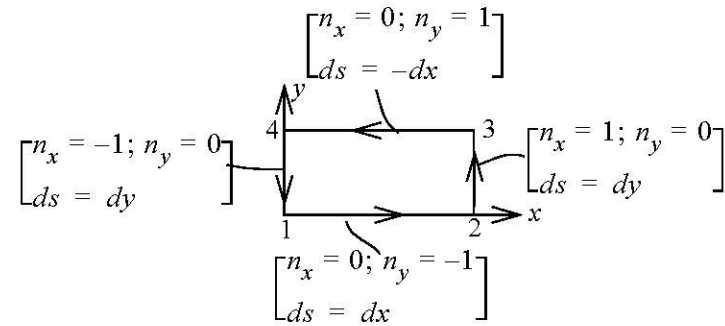
variation of primary variables. Thus, stationary value of a functional implies that at each boundary point either the secondary variable is specified or the primary variable be specified (zero or non-zero).

$$I = \int_a^b F(u^{(r)}, u^{(r-1)}, u^{(r-1)}, \quad \cdot \quad \cdot \quad \cdot \quad , u^{(1)}, u^{(0)}, x) \ dx$$

Where $u^{(r)} = d^r u / dx^r$. The stationary value $\delta I = 0$ implies the following

The differential equation is: $\dfrac{\partial F}{\partial u^{(0)}} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial u^{(1)}}\right) + \dfrac{d^2}{dx^2}\left(\dfrac{\partial F}{\partial u^{(2)}}\right) - \dfrac{d^3}{dx^3}\left(\dfrac{\partial F}{\partial u^{(3)}}\right) + \quad \cdot \quad \cdot \quad \cdot \quad + (-1)^r \dfrac{d^r}{dx^r}\left(\dfrac{\partial F}{\partial u^{(r)}}\right) = 0$

All possible boundary conditions are:

Natural Boundary Conditions	Essential Boundary Conditions.
$-\left(\dfrac{\partial F}{\partial u^{(1)}}\right) + \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial u^{(2)}}\right) - \dfrac{d^2}{dx^2}\left(\dfrac{\partial F}{\partial u^{(3)}}\right) + \bullet \bullet + (-1)^r \dfrac{d^{(r-1)}}{dx^{(r-1)}}\left(\dfrac{\partial F}{\partial u^{(r)}}\right) = 0$	$\delta u = 0$
$\left(\dfrac{\partial F}{\partial u^{(2)}}\right) - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial u^{(3)}}\right) + \bullet \bullet + (-1)^r \dfrac{d^{(r-2)}}{dx^{(r-2)}}\left(\dfrac{\partial F}{\partial u^{(r)}}\right) = 0$	$\delta u^{(1)} = 0$
$-\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial u^{(3)}}\right) + \bullet \bullet + (-1)^r \dfrac{d^{(r-3)}}{dx^{(r-3)}}\left(\dfrac{\partial F}{\partial u^{(r)}}\right) = 0$	$\delta u^{(2)} = 0$
$\bullet = 0$	$= 0$
$\bullet = 0$	$= 0$
$\bullet = 0$	$= 0$
$\dfrac{\partial F}{\partial u^{(r)}} = 0$	$\delta u^{(r-1)} = 0$

- If the functional I has the highest derivative of order r then the differential equation will be of order $2r$.
- For differential equations to be linear the functional can at most be quadratic in u and its derivatives.
- The boundary term with the symbol $\boldsymbol{\delta}$ are the **primary variables**. The boundary conditions on the primary variables are called **essential boundary condition**.Primary variables are the (r-1) derivatives. These derivatives are

also called **principal derivatives**.

- The term multiplying the primary variables are the **secondary variables**. The boundary conditions on the secondary variables are called **natural boundary conditions**.

- If the functional contains more than 1 variable (u, v, w) then the above equations have to be written with respect to each of the variables.

Stationary value of an area integral:

$$I = \iint\limits_{A} F(u, u_x, u_y, x, y) \, dx \, dy$$

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0 \qquad x, y \text{ in } A$$

$$\frac{\partial F}{\partial u_x} n_x + \frac{\partial F}{\partial u_y} n_y = 0 \qquad or \qquad \delta u = 0 \qquad x, y \text{ on } \Gamma$$

Stationary value of a volume integral:

$$I = \iint\limits_{T}\int F(u, u_x, u_y, u_z, x, y, z) \, dx \, dy \, dz$$

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) = 0 \qquad x, y, z \text{ in } T$$

$$\frac{\partial F}{\partial u_x} n_x + \frac{\partial F}{\partial u_y} n_y + \frac{\partial F}{\partial u_z} n_z = 0 \qquad or \qquad \delta u = 0 \qquad x, y, z \text{ on } S$$

- Generalization to higher order derivatives will have form similar to 1-D with partial derivatives in each direction in place of ordinary derivatives. There are however terms related to curvatures and corners that will come in the natural boundary conditions.

Linear Functional $l(u)$

$$l(\alpha u) = \alpha l(u) \qquad l(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 l(u_1) + \alpha_2 l(u_2)$$

Bilinear Functional $B(u, v)$

$$B(\alpha u + \beta v) = \alpha B(u) + \beta B(v)$$

$$B(\alpha_1 u_1 + \alpha_2 u_2, \beta_1 v_1 + \beta_2 v_2) = \alpha_1[\beta_1 B(u_1, v_1) + \beta_2 B(u_1, v_2)] + \alpha_2[\beta_1 B(u_2, v_1) + \beta_2 B(u_2, v_2)]$$

$$B(u, v) = B(v, u) \qquad \text{Symmetric Bi-linear Functional}$$

Approximate Methods

Approximation: $u = \phi_o + \sum_{j=1}^{n} c_j \phi_j$

ϕ_j is a set of complete and independent approximating functions. On substituting in the boundary value problem we obtain the errors:

e_d— Error in differential equation.

e_n— Error in natural boundary condition.

e_e— Error in essential boundary condition.

Commonalities

- Produce a set of algebraic equations in the unknown constants c_j.
- Choose ϕ_i to set one (or two) of the errors e_d, e_e, or e_n to zero
- Minimize the remaining error.

Differences

- Which error is set to zero

  Domain Methods: $e_e = 0$ or $e_n = 0$

        FEM-Stiffness version: $e_e = 0$

        FEM-Flexibility version: $e_n = 0$

  Boundary Methods: $e_d = 0$

- Error Minimizing Process

Various Methods

Description below is in context of domain methods but these can also be used for boundary methods. Thus, ϕ_j must

be differentiable up to the order of differential equation. ϕ_o is chosen to satisfy all non-homogeneous part of ALL boundary conditions. ϕ_i are chosen to satisfy homogeneous part of ALL boundary conditions.

Weighted Residue (Petro-Galerkin)

$$\int_V \psi_k e_d dV = 0$$

Makes the error orthogonal to the weighting functions ψ_k. Very general but depends upon the choice of ψ_k

Collocation

$$e_d(x_k) = 0$$

Simplest method. General but depends upon location of x_k

Least Square

$$E = \int_V e_d^2 dV \qquad \frac{\partial E}{\partial c_k} = 0$$

It is general. Minimizes error. Produces a symmetric matrix. Can be computationally very expensive.

Galerkin: In weighted residue choose: $\psi_k = \phi_k$

$$\int_V \phi_k e_d dV = 0$$

Rayleigh Ritz's Method

Functional: $I(u) = \frac{1}{2}B(u, u) - l(u)$

Approximation: $u = \phi_o + \sum_{i=1}^{n} c_i \phi_i$

1. ϕ_o is chosen to satisfy all non-homogeneous part of ALL *essential* boundary conditions. ϕ_i are chosen to

satisfy homogeneous part of ALL *essential* boundary.

2. ϕ_i must be differentiable at least to the highest order of derivative in the functional I(u).

3. The set of ϕ_i must be independent.

4. The set of ϕ_i must be complete.

Algebraic Equations: $[K]\{c\} = \{R\}$

$$K_{kj} = \frac{1}{2}[B(\phi_k, \phi_j) + B(\phi_j, \phi_k)] \qquad R_k = l(\phi_k) - \frac{1}{2}[B(\phi_k, \phi_o) + B(\phi_o, \phi_k)]$$

If $B(u, u) > 0$ for any u then:

(a) the matrix $[K]$ will be positive definite.

(b) $I(u_N) \geq I(u_M) \qquad N \leq M$

Weak Form

The objective is to weaken the continuity requirements on ϕ_i to half from the order of derivative of the differential equation. (similar to Ritz's requirement).

Weak form may be obtained in one of the two ways:

1. Taking the variation of the functional and not transferring any derivatives. (Requires existence of a functional.)

2. Multiplying the differential equation by a function (v = δu) and transferring half the derivatives from u on to v and substituting the boundary conditions.

Finite Element Method

Polynomial Approximations

- If a variable (coordinates or primary variables) is modeled using a higher order polynomial than it actually is, then it will be modeled exactly as long as the set of approximating functions are complete.
- Continuity of a variable on the element boundary is assured if the variable depends only upon the nodes on the element boundary.

Lagrange Polynomials

1. Ensures continuity of the function at end nodes. Derivatives of the function can be discontinuous.

2. Its value is one at its own node and zero at other nodes.

$$\phi_i(P_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad P_j \text{ are coordinates of node j.}$$

Multiply the equations of node (1-D), or equations of line containing the nodes (2-D), or equations of planes containing the nodes (3-D) to ensure 0 value at other nodes. Divide the product by substituting the coordinates of its own node to get value of 1.

3. $\sum_{i=1}^{n} \phi_i = 1$

4. Lagrange Polynomials are used in approximation of 2nd order differential equations.

Hermite Polynomials

1. Ensures continuity of the function and derivatives at end nodes.

2. It is zero if the derivative order or the node are different. It is one if derivative order and the node is the same.

$$\frac{d^k H_n^{\{m\}}}{d\xi^k}(\xi_p) = \begin{cases} 0 & n \neq p & or & k \neq m \\ 1 & n = p & and & k = m \end{cases}$$
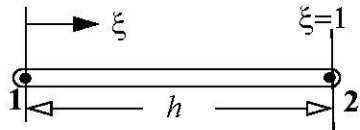
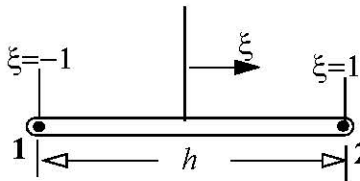3. Hermite polynomials are used for 4th and higher order differential equations.

Natural Coordinates

1. Coordinates which vary between 0 and 1 or -1 and 1.

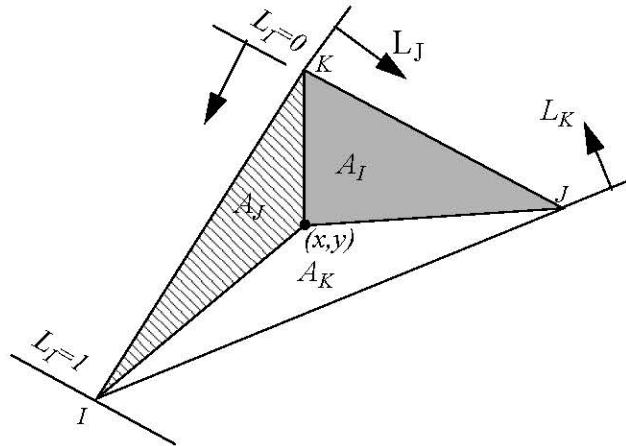2. Natural coordinates are non-dimensional coordinates.

1-d Natural Coordinates

$$\xi = \left[\frac{x - x_1}{x_2 - x_1}\right] \qquad \frac{d\phi_i}{dx} = \left(\frac{1}{h}\right)\frac{d\phi_i}{d\xi} \qquad dx = |J|d\xi = hd\xi$$

$$\xi = 2\left[\frac{x - (x_2 + x_1)/2}{x_2 - x_1}\right] \qquad \frac{d\phi_i}{dx} = \left(\frac{2}{h}\right)\frac{d\phi_i}{d\xi} \qquad dx = |J|d\xi = \frac{h}{2}d\xi$$

2-d Natural Coordinates

Area Coordinates

$$L_I = \frac{1}{2A}[(x - x_K)(y_J - y_K) - (y - y_K)(x_J - x_K)]$$

$$A = \frac{1}{2}[(x_I - x_K)(y_J - y_K) - (y_I - y_K)(x_J - x_K)]$$
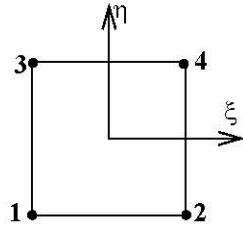
$$L_I = \frac{A_I}{A} \qquad L_J = \frac{A_J}{A} \qquad L_K = \frac{A_K}{A} \qquad L_I + L_J + L_K = 1$$

$$\frac{\partial\phi_i}{\partial x} = \left(\frac{\partial\phi_i}{\partial L_I}\right)\frac{\partial L_I}{\partial x} + \left(\frac{\partial\phi_i}{\partial L_J}\right)\frac{\partial L_J}{\partial x} + \left(\frac{\partial\phi_i}{\partial L_K}\right)\frac{\partial L_K}{\partial x}$$

$$\int_a^b L_I^m L_J^n L_K^p \ ds = (b - a)\frac{m!n!p!}{(m + n + p + 1)!} \qquad \iint_A L_I^m L_J^n L_K^p \ dxdy = (2A)\frac{m!n!p!}{(m + n + p + 2)!}$$
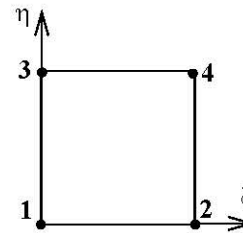
Use Pascal Triangle to determine the number of nodes on a triangular element.

Rectangular Elements

$$-1 \le \xi \le 1 \qquad -1 \le \eta \le 1$$

$$0 \le \xi \le 1 \qquad 0 \le \eta \le 1$$

$$\xi = 2\left[\frac{x - (x_2 + x_1)/2}{x_2 - x_1}\right] \qquad \eta = 2\left[\frac{y - (y_3 + y_1)/2}{y_3 - y_1}\right]$$
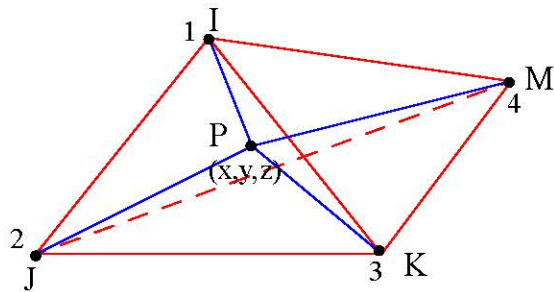
$$\xi = \left[\frac{x - x_1}{x_2 - x_1}\right] \qquad \eta = \left[\frac{y - y_1}{y_3 - y_1}\right]$$

• Use tensor product to obtain the Lagrange polynomials for higher order.

Serendipity Elements: No nodes in the interior.

3-d Natural Coordinates

Volume Coordinates

$$L_I = \frac{\text{Volume PJKM}}{V} = \frac{V_I}{V}$$

$$L_J = \frac{\text{Volume PKMI}}{V} = \frac{V_J}{V}$$

$$L_K = \frac{\text{Volume PMIJ}}{V} = \frac{V_K}{V}$$

V = Volume of the tetrahedron IJKM=V_I+V_J+V_K+V_M

$$L_I + L_J + L_K + L_M = 1$$

$$L_M = \frac{\text{Volume PIJK}}{V} = \frac{V_M}{V}$$

$$L_I = \frac{1}{6V}[(x - x_J)(y_{KJ}z_{MJ} - z_{KJ}y_{MJ}) + (y - y_J)(z_{KJ}x_{MJ} - x_{MJ}z_{KJ}) + (z - z_J)(x_{KJ}y_{MJ} - y_{KJ}x_{MJ})]$$
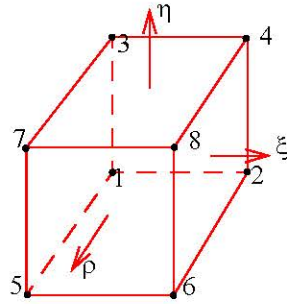
$$V = \frac{1}{6}[(x_I - x_J)(y_{KJ}z_{MJ} - z_{KJ}y_{MJ}) + (y_I - y_J)(z_{KJ}x_{MJ} - x_{MJ}z_{KJ}) + (z_I - z_J)(x_{KJ}y_{MJ} - y_{KJ}x_{MJ})]$$

$$\frac{\partial \phi_i}{\partial x} = \left(\frac{\partial \phi_i}{\partial L_I}\right)\frac{\partial L_I}{\partial x} + \left(\frac{\partial \phi_i}{\partial L_J}\right)\frac{\partial L_J}{\partial x} + \left(\frac{\partial \phi_i}{\partial L_K}\right)\frac{\partial L_K}{\partial x} + \left(\frac{\partial \phi_i}{\partial L_M}\right)\frac{\partial L_M}{\partial x}$$

$$\iiint_V L_I^m L_J^n L_K^p L_M^q \, dx\, dy\, dz = (6V)\frac{m!\, n!\, p!\, q!}{(m+n+p+q+3)!}$$

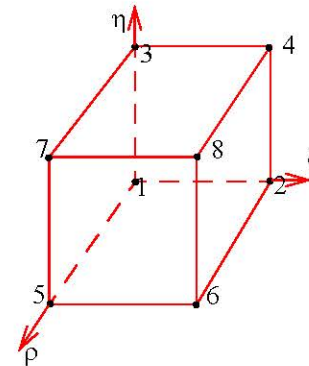- Use Pascal tetrahedron to determine the number of nodes needed for higher order polynomials.

Brick Elements

$$-1 \leq \xi \leq 1 \qquad -1 \leq \eta \leq 1 \qquad -1 \leq \rho \leq 1$$

$$\xi = 2\left[\frac{x - (x_2 + x_1)/2}{x_2 - x_1}\right] \qquad \eta = 2\left[\frac{y - (y_3 + y_1)/2}{y_3 - y_1}\right]$$

$$\rho = 2\left[\frac{z - (z_5 + z_1)/2}{z_5 - z_1}\right]$$

$$0 \leq \xi \leq 1 \qquad 0 \leq \eta \leq 1 \qquad 0 \leq \rho \leq 1$$

$$\xi = \left[\frac{x - x_1}{x_2 - x_1}\right] \qquad \eta = \left[\frac{y - y_1}{y_3 - y_1}\right]$$

$$\rho = \left[\frac{z - z_1}{z_5 - z_1}\right]$$

Serendipity Elements: No nodes either on the surface or in the interior. The nodes are only along the edges.

Conforming and Non-conforming Elements

Conforming Element: All continuity requirements are met on the element boundary.

Non-Conforming Element: All continuity requirements are met at the element nodes but the continuity requirement on the element boundary is relaxed.

• Non-conforming elements may only converge in a particular orientation.

• Non-conforming elements are used because fully conforming element
—requires too many degrees of freedom.
—may result in continuity of secondary variable even when a discontinuity is needed.
—a given direction may not require the same accuracy.

Coordinate Transformation

Let the order of polynomial for approximation: of coordinates (geometry) be M and of primary variables be N.

Subparametric: $M < N$

Isoparametric: $M = N$

Superparametric: $M > N$.

Original Element: The element of the mesh that can have curved boundaries.

Master Element: The element in natural coordinates for which approximating functions are created.

Jacobian matrix: Matrix relating derivatives in natural coordinates to derivatives in original coordinates (Cartesian).

Jacobian: Determinant of the Jacobian matrix. Used in changing the original integrating variables to natural variables.

Procedural steps in the finite element method

Step 1:   Identify discontinuities in the secondary variables, material properties, and source terms.

Step 2:   Divide the body into elements, making sure that the discontinuities are on the element boundaries.

Step 3:   Construct the weak form at the element level. Identify the bilinear and linear functional.

Step 4:   Construct the approximating function for the primary variables (ϕ_i)

Step 5:   Develop the coordinate transformation equations that convert the Cartesian coordinates to natural coordinates. Compute the Jacobian matrix and its inverse.

Step 6:   Obtain the element matrices and the element right hand side vectors. Assume B(u,v)=B(v,u)

$$K_{kj}^{(e)} = B^{(e)}(\phi_k, \phi_j) \qquad R_k^{(e)} = l(\phi_k)$$

Step 7:   Transform from local orientation to global orientation if needed.

$$[K_G^{(e)}] = [T]^T[K^{(e)}][T] \qquad [R_G^{(e)}] = [T]^T[R^{(e)}]$$

Step 8:   Assemble the global matrix and global right hand side vector.

$$\delta I = \sum_e \delta I^{(e)} = \sum_e \delta I^{(e)} = \sum_e \left\{\delta u_G^{(e)}\right\}^T \left([b_G^{(e)}]\left\{u_G^{(e)}\right\} - \left\{R_G^{(e)}\right\}\right)$$

Step 9:   Draw the equivalence for the secondary variables.

Step 10:   Incorporate the boundary conditions.

$$\delta I = \{\delta u_G\}^T([K_G]\{u_G\} - \{R_G\}) = 0$$

Step 11:   Solve the algebraic equations for the primary variables. $[K_G]\{u_G\} = \{R_G\}$

Step 12:   Obtain secondary variables and other quantities of interest.

Step 13:   Interpret and check the results.

Step 14:   Refine the mesh if necessary, and repeat the above steps.

Numerical Integration

1-D: $I = \int_{-1}^{1} F(\xi)d\xi = \sum_{i=1}^{n} w_i F(\xi_i)$

- w_i are called the weights; ξ_i are called the base points; and n is number of base points.Functions are evaluated at base points.

2-D: $I = \int_{-1}^{1} \int_{-1}^{1} F(\xi, \eta)(d\xi)d\eta = \sum_{j=1}^{n} \sum_{i=1}^{n} w_i w_j F(\xi_i, \eta_j)$

3-D: $I = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} F(\xi, \eta, \rho)d\xi d\eta d\rho = \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} w_i w_j w_k F(\xi_i, \eta_j, \rho_k)$

- Secondary variables and related quantities are evaluated at Gauss (base) points.

Storage and Solution

Banded Matrix: All non-zero elements of the global matrix are within the bandwidth ($2N_B$). The bandwidth is dictated by the difference between the highest and the lowest node number on the elements. Stored in rectangular matrix of N x N_B.

Skyline Storage: Used for sparse matrices. Element stored as vector composed of segments of H_i, where H_i is the column height above the diagonal containing the highest non-zero element in the global matrix.

Wavefront equation solver: Assembly and equation solving proceeds simultaneously. Wavefront number is a measure of number of coefficients being manipulated in the solution process at a given step.

Substructure (Matrix Partitioning): Internal nodes of an element or a structure are eliminated in terms of the node on the boundary.

Direct Methods of Equation Solving: Good for multiple load vectors. Matrix inversion, Gauss Elimination, Gauss-Jordan, Cholesky's Method.

Iterative Methods: Good for time dependent and non-linear problems. Jacobi, Gauss-Seidel, Successive Overrelaxation.

Gauss Elimination: $[K]=[L][U]$; Solves the solution by forward and backward substitution.

Cholesky's Method: Can be used only for positive definite symmetric matrix. $[K]=[L][L]^T$

Modeling

- A model is a symbolic representation of the real thing.

Errors in FEM

   1. Modeling error:

Error that arise from the description of the boundary value problem (BVP): Geometric description, material description, loading, boundary conditions, type of analysis.

Non-linear analysis: Material non-linearity; Geometric non-linearity; Contact problem.

2. Discretization error

Errors that arises from creation of the mesh.

Mesh refinement methods: h-method; p-method; r-method and their combinations.

Error norms: L_1 norm; L_2 norm; Energy norms.

3. Numerical error

Errors that arise from finite digit arithmetic and use of numerical methods.

Integration error; Round off error; Matrix conditioning error.

Time Dependent Problems

1. Treat time like any other space variable and obtain the weak form by integrating over time and space.

$$u = \sum_{i=1}^{n} u_i^{(e)} \phi_i(x, y, z, t)$$

2. Assume that time and space can be separated. Weak form is constructed by integrating over space and we obtain ordinary differential equations in time.

$$u = \sum_{i=1}^{n} u_i^{(e)}(t) \phi_i(x, y, z)$$

$$[M]\left\{\frac{d^2 u}{dt^2}\right\} + [C]\left\{\frac{du}{dt}\right\} + [K]\{u\} = \{R\}$$

- The matrices $[M]$ and $[K]$ are symmetric, positive definite matrices. The matrix $[C]$ may not be positive definite but is symmetric.
- Eigenvectors are orthogonal with respect to $[M]$, $[K]$, and $[C]$.

Classification of problems

1. Free response: {R}={0}--eigenvalue problem.

Numerical Methods: $[H]\{y\} = \lambda\{y\}$. Symmetric matrix preferred: $[H] = [H]^T$

Free vibrations; 2nd order equations: [C]=0 and {R}=0. $\{u\} = \{x\}e^{\bar{i}\omega t}$

Consistent mass matrix. $[K] = [L][L]^T$

Lumped mass matrix. $[M] = [M]^{\frac{1}{2}}[M]^{\frac{1}{2}}$

First order equations: {R}=0 and {M}=0 $\quad \{u\} = \{x\}e^{-\omega t} \qquad\qquad [K] = [L][L]^T$

Free response of a damped systems: {R}=0 $\quad \{u\} = \{x\}e^{\omega t} \quad \omega = \omega_1 + \bar{i}\,\omega_2$

Rayleigh damping: $\qquad [C] = \alpha[M] + \beta[K]$

2. Periodic response: $\{R(t)\}$ is periodic

$$\{R\} = \{f\}e^{\omega t} \qquad\qquad \{u\} = \{x\}e^{\omega t} \quad \omega = \omega_1 + \bar{i}\,\omega_2$$

3. Transient (forced) response: $\{R(t)\}$ is arbitrary.

Indirect Method: Frequency response:

Any function can be represented by a Fourier series

$$\{R\} = \sum_{j=1}^{n} \{f_j\}e^{\bar{i}\,\omega_j t} \qquad \{u\} = \sum_{j=1}^{n} \{x_j\}e^{\bar{i}\,\omega_j t}$$

Indirect Method: Modal analysis:

The response of a system is a linear combination of the eigenvectors of the system. The basic idea here is to find the coefficients of the linear combination.

$$\{u_{free}\} = \sum_{j=1}^{n} \{x_j\}e^{\omega_j t} \qquad \{u\} = \sum_{j=1}^{n} y_j(t)\{x_j\}$$

$$m_i\frac{d^2 y_i}{dt^2} + c_i\frac{dy_i}{dt} + k_i y_i = r_i \qquad i = 1 \text{ to n}$$

Direct Method: First order system

Approximate time derivatives by finite difference.

$$\theta\left\{\frac{du}{dt}\right\} + (1-\theta)\left\{\frac{du}{dt}\right\}_{n+1} = \frac{\{u\}_{n+1} - \{u\}_n}{\Delta t_{n+1}}$$

Solution at time $t = t_{n+1}$ is found from solution at time $t = t_n$.