# Numerical Integration (Quadrature)

## 1-D Numerical integration

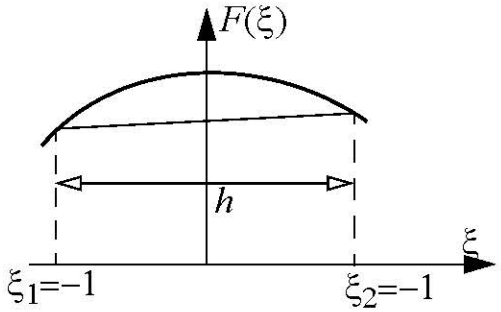
$$I = \int_{-1}^{1} F(\xi)d\xi = \sum_{i=1}^{n} w_i F(\xi_i)$$

- $w_i$ are called the weights; $\xi_i$ are called the base points; and n is number of base points.Functions are evaluated at base points.

## Trapezoidal rule: ($n$ = 2, linear approximation of $F(\xi)$

$$I = \int_{-1}^{1} F(\xi)d\xi = \frac{h}{2}(F(\xi_1) + F(\xi_2))$$

$$w_1 = w_2 = \frac{h}{2}$$




## Simpson's rule ($n$ = 3, quadratic approximation of $F(\xi)$):

$$I = \int_{-1}^{1} F(\xi)d\xi = \frac{h}{3}(F(\xi_1) + 4F(\xi_2) + F(\xi_3))$$

## Newton-Cotes quadrature

- Equally spaced base points.

For $(n-1)^{th}$ order polynomial is integrated exactly by choosing $n$ base points. Hence the error is of the order of $h^{(n+1)}$, where $h$ is the distance between two data points. Note there are $n$ parameters (weights) that can be adjusted.

## Gauss quadrature (Gauss-Legendre quadrature)

Location of base points $\xi_i$ is determined such that the error is minimized. Thus, there are $2n$ parameters. Hence, a polynomial of order $(2n-1)$ will be integrated exactly. Using this idea the weights and base points are determined. It turns out that the base points are the roots of Legendre Polynomials.

**Table 1: Weights and Gauss (base) points**

| n | Base points $\xi_i$ | Weights $w_i$ |
|---|---|---|
| One point formula | 0.0 | $w_1 = 2.0$ |
| Two point formula | $\pm(1/\sqrt{3}) = \pm 0.5774$ | $w_1 = w_2 = 1.0$ |
| Three point formula | $0.0; \pm\sqrt{0.6} = \pm 0.7746$ | $w_1 = (8/9); w_2 = w_3 = (5/9)$ |

## 2-D Numerical integration

$$I = \int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta)\,d\xi\,d\eta = \sum_{j=1}^{n}\sum_{i=1}^{n} w_i w_j F(\xi_i,\eta_j)$$

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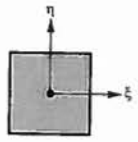
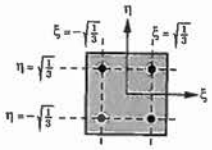
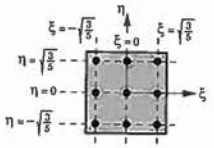
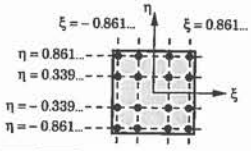
**Table 9.3.1** Selection of the integration order and location of the Gauss points for linear, quadratic, and cubic quadrilateral elements (nodes not shown).

| Element type | Maximum polynomial degree | Order of integration ($r \times r$) | Order of the residual | Location of integration points* in master element |
|---|---|---|---|---|
| Constant ($r = 1$) | 0 | $1 \times 1$ | $O(h^2)$ |  |
| Linear ($r = 2$) | 2 | $2 \times 2$ | $O(h^4)$ |  |
| Quadratic ($r = 3$) | 4 | $(3 \times 3)$ | $O(h^6)$ |  |
| Cubic ($r = 4$) | 6 | $(4 \times 4)$ | $O(h^8)$ |  |

*See Table 6.1.2 for the integration points and weights for each coordinate direction.

## 3-D Numerical integration

$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta,\rho)\,d\xi\,d\eta\,d\rho = \sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} w_i w_j w_k F(\xi_i,\eta_j,\rho_k)$$

C.1 Evaluate the integral below using 1,2, and 3 points Gauss Quadrature and compare with analytical value.

$$I = \int_{-1}^{1}\int_{-1}^{1} \frac{(\xi^2 + \xi + 2)}{1 + \eta^2} d\xi d\eta$$

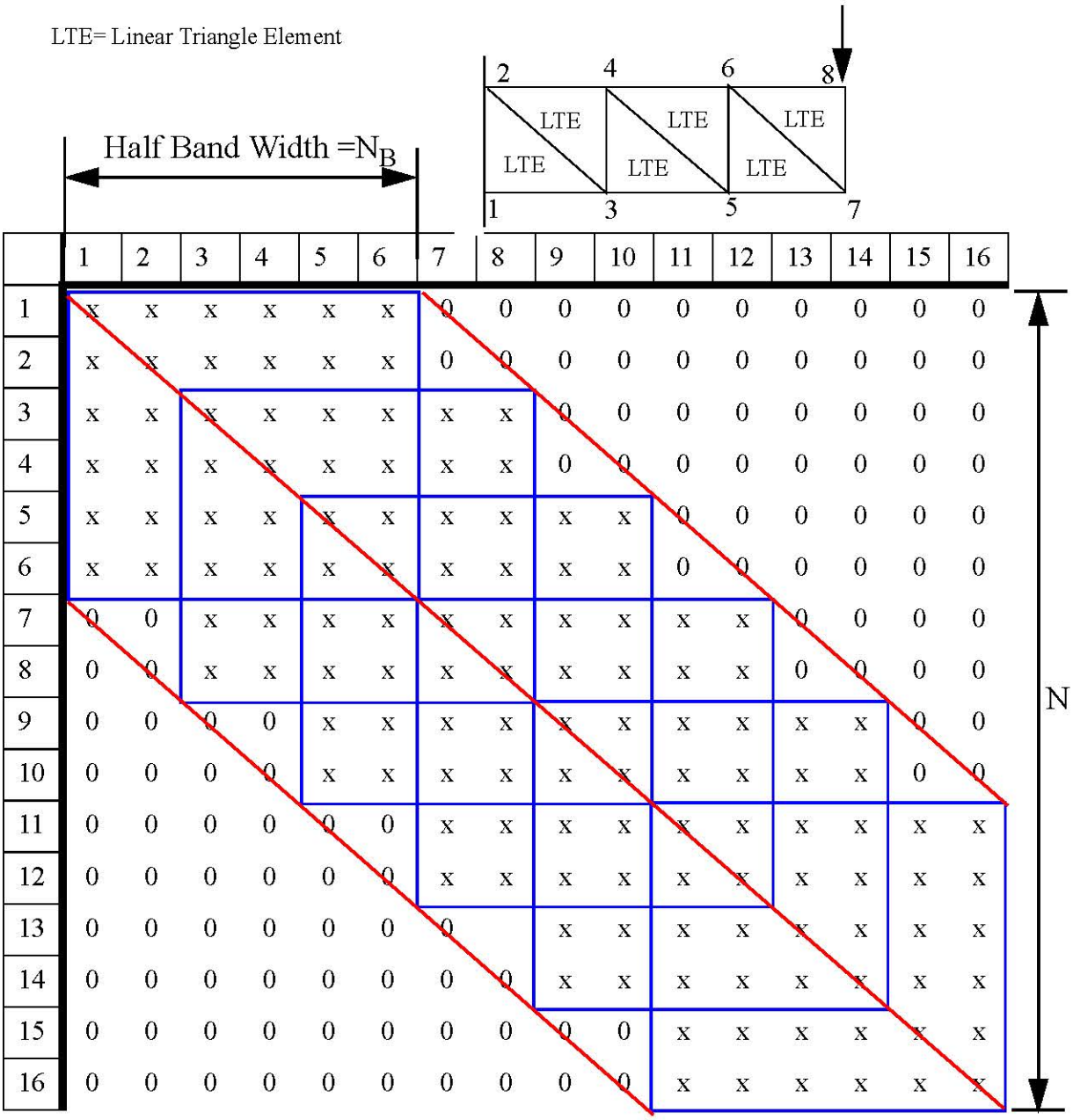
- More Gauss points increases the accuracy of integration but may make the element more stiff thus decreasing the FEM accuracy.
- Two few Gauss points may cause instabilities resulting in singular stiffness matrices.
- Evaluation of secondary variables and related quantities (stresses) at Gauss points is usually more accurate than at nodes.

# Storage and Solution Techniques

## Banded Matrix

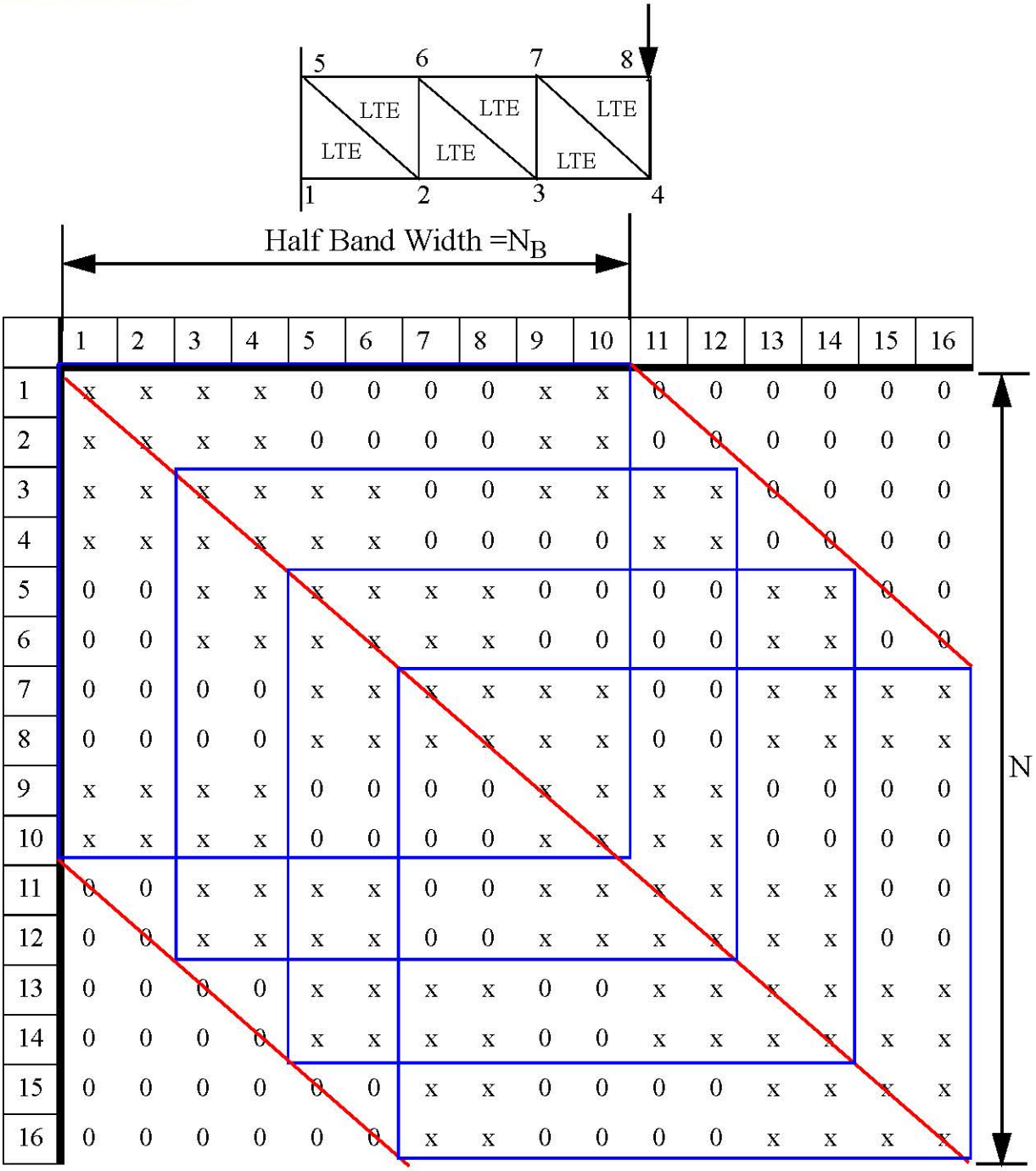
- Each node has two degrees of freedom (u and v)
- Element matrix is 6 x 6

LTE= Linear Triangle Element




- Only the upper banded form of the global stiffness matrix is stored in a rectangular matrix of size $NxN_B$

## Banded Matrix




| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | x | x | x | x | 0 | 0 | 0 | 0 | x | x | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | x | x | x | x | 0 | 0 | 0 | 0 | x | x | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | x | x | x | x | x | x | 0 | 0 | x | x | x | x | 0 | 0 | 0 | 0 |
| 4 | x | x | x | x | x | x | 0 | 0 | 0 | 0 | x | x | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | x | x | x | x | x | x | 0 | 0 | 0 | 0 | x | x | 0 | 0 |
| 6 | 0 | 0 | x | x | x | x | x | x | 0 | 0 | 0 | 0 | x | x | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | x | x | x | x | x | x | 0 | 0 | x | x | x | x |
| 8 | 0 | 0 | 0 | 0 | x | x | x | x | x | x | 0 | 0 | x | x | x | x |
| 9 | x | x | x | x | 0 | 0 | 0 | 0 | x | x | x | x | 0 | 0 | 0 | 0 |
| 10 | x | x | x | x | 0 | 0 | 0 | 0 | x | x | x | x | 0 | 0 | 0 | 0 |
| 11 | 0 | 0 | x | x | x | x | 0 | 0 | x | x | x | x | x | x | 0 | 0 |
| 12 | 0 | 0 | x | x | x | x | 0 | 0 | x | x | x | x | x | x | 0 | 0 |
| 13 | 0 | 0 | 0 | 0 | x | x | x | x | 0 | 0 | x | x | x | x | x | x |
| 14 | 0 | 0 | 0 | 0 | x | x | x | x | 0 | 0 | x | x | x | x | x | x |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | x | x | 0 | 0 | 0 | 0 | x | x | x | x |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | x | x | 0 | 0 | 0 | 0 | x | x | x | x |

- The bandwidth is solely dictated by the difference between the highest and the lowest node number on the elements.

# Equation Solving [K]{d}={R}

Direct Methods: Matrix inversion, Gauss Elimination, Gauss-Jordan, Cholesky's Method.
Iterative Methods: Jacobi, Gauss-Seidel, Successive Overrelaxation.

## Gauss Elimination

- A lower triangular matrix has zero coefficients above the diagonal [L]
- A upper triangular matrix has zero coefficients below the diagonal [U]

Any matrix [K] can be decomposed as [K]=[L][U]. The algebraic equations become: [L][U]{d}={R}. If we substitute [U]{d}={x} we obtain [L]{x}={R}. The solution is obtained in two steps:

1. Forward substitution: Solve for {x} from: [L]{x}={R}
2. Backward substitution: Solve for {d} from: [U]{d}={x}

- If the algebraic matrix is symmetric and positive definite then the decomposition is done using Cholesky's method to obtain $[U]=[L]^T$.
- Forward substitution is done in increasing node number order and backward substitution is done in decreasing node number order.

## Pros and Cons

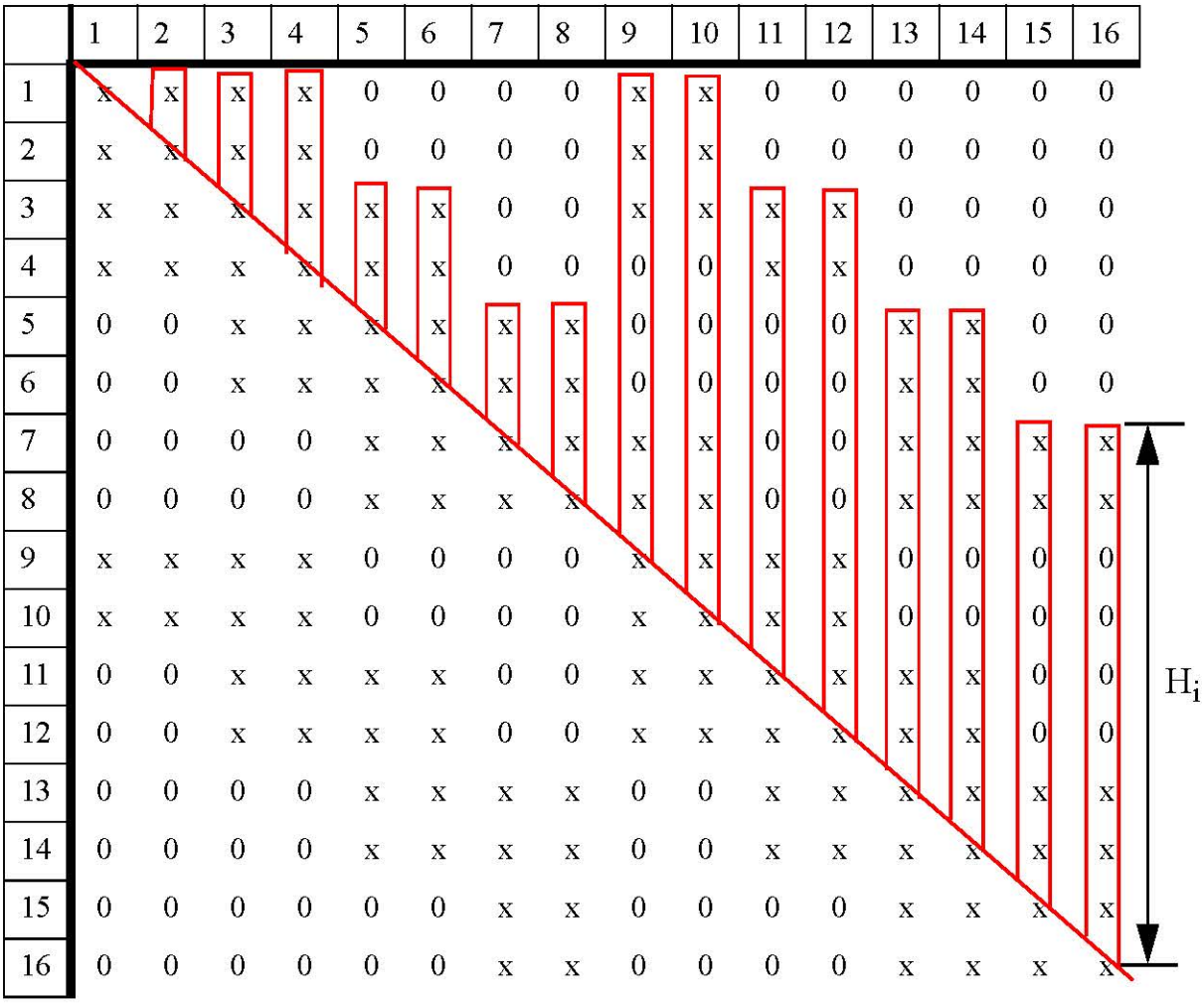
- Direct methods are better for multiple load vectors.The computation time is proportional to $NN_B^2$ .
- Computation times for iterative methods depend upon the starting guess solution. In time dependent problems that are solved in small time steps the initial guess vector is known from previous time step. Non-linear problems which are solved as linear problems iteratively also use less time in iterative methods.

# Large Systems

## Skyline storage scheme

The stiffness matrix of very large structures may have mostly zero terms, even inside the band width. Such matrices are called sparse matrices.

$H_i$= The height of the column above the diagonal of the $i^{th}$ row beyond which there is no non-zero term.




Skyline algorithms are used for sparse matrices.

## Wavefront (Frontal) equation solver:

- Assembly and equation solving proceeds simultaneously.
- Computation of coefficients in a stiffness matrix and load vector is complete only when contribution of all elements that share the node have been added.
- A complete coefficient can be processed in the solution process.
- Wavefront number is a measure of number of coefficients being manipulated in the solution process at a given step.

## Substructure (Matrix partitioning)

Algebraic equations can be written as:

$$\left[\begin{array}{c|c} [A] & [B] \\ \hline [C] & [D] \end{array}\right] \left\{\begin{array}{c} \{d_1\} \\ \hline \{d_2\} \end{array}\right\} = \left\{\begin{array}{c} \{R_1\} \\ \hline \{R_2\} \end{array}\right\}$$

The two sets of matrix equations can be solved as shown below.

$$[A]\{d_1\} + [B]\{d_2\} = \{R_1\} \text{ or } \quad \{d_1\} = [A]^{-1}(\{R_1\} - [B]\{d_2\})$$

$$[C]\{d_1\} + [D]\{d_2\} = \{R_2\}$$

$$[C][A]^{-1}(\{R_1\} - [B]\{d_2\}) + [D]\{d_2\} = \{R_2\}$$

$$([D] - [C][A]^{-1}[B])\{d_2\} = \{R_2\} - [C][A]^{-1}\{R_1\}$$

$$[K_S]\{d_2\} = \{R_S\}$$

In substructuring $\{d_2\}$ are the nodal displacements on the interface of two substructures and $\{d_1\}$are the displacements of nodes on either of the substructure.

- Substructuring is used for large structures. Each substructure can be used as a superelement if a substructure repeats in a structure.
- Substructuring could be used for creating meshes with different densities either for creating original mesh or for mesh refinements.
- In non-linear analysis such as in elastic-plastic analysis. The plastic zone is made a part of a substructure.