Indicial Notation

The learning objectives in this chapter are:

- Understand indicial notation and the rules of manipulating the indices.
- Understand the applications of indicial notation to mechanics of materials equations.

Definitions and Concepts

Summation Convention

A force vector and displacement vector in Cartesian coordinates may be written as $\vec{F} = \{F_x, F_y, F_z\}$ and $\overrightarrow{\Delta u} = \{\Delta u_x, \Delta u_y, \Delta u_z\}$. The work done ($W$) by the force $\vec{F}$ in moving a point by an amount $\overrightarrow{\Delta u}$ is given by the scalar (dot) product shown below.

$$W = \vec{F} \bullet \overrightarrow{\Delta u} = F_x \Delta u_x + F_y \Delta u_y + F_z \Delta u_z = \sum_{i=1}^{3} F_i \Delta u_i$$

- *repeated index implies summation* unless it is explicitly stated otherwise. $W = F_i \Delta u_i$
- *free indices in every term of an equation must be the same* unless explicitly stated that an index is not free and is being used symbolically.
- A free index can be replaced by another free index and a dummy index can be replaced by another dummy index.

Kronecker delta function (or just delta function)

$$\boxed{\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}}$$

$$\delta_{ij} F_j = F_i \qquad \delta_{ij} A_{jk} = A_{ik} \qquad \delta_{ij} A_{jkm} = A_{ikm}$$

$$\boxed{\delta_{ii} = 2 \quad \text{in two dimension} \quad \text{and} \quad \delta_{ii} = 3 \quad \text{in three dimension}}$$

$$\boxed{\delta_{jk} A_{jk} = 0 \qquad \text{IF } A_{jk} = -A_{kj}}$$

- The product of the delta function with a function that is anti-symmetric in its subscript will result in a zero value.

Permutation function

$$e_{ijk} = \begin{cases} 0 & \text{if any } i, j, \text{ or } k \text{ are same,} \\ 1 & \text{if } ijk \text{ are in cyclic order,} \\ -1 & \text{if } ijk \text{ are noncyclic order.} \end{cases} \quad \text{in three dimension}$$

The moment vector $\vec{M} = \{M_1, M_2, M_3\}$ due to the force at a point from which a position vector is drawn $\vec{r} = \{r_x, r_y, r_z\}$ is given by the vector (cross) product as

$$\vec{M} = \vec{r} \times \vec{F} = \{r_y F_z - r_z F_y, r_z F_x - r_x F_z, r_x F_y - r_y F_x\} \qquad M_1 = r_2 F_3 - r_3 F_1 \qquad M_2 = r_3 F_1 - r_1 F_3 \qquad M_3 = r_1 F_2 - r_2 F_1$$

$$M_i = e_{ijk} r_j F_k$$

$$e_{ij} = \begin{cases} 0 & \text{if any } i = j \\ 1 & \text{if } ij \text{ are in cyclic order,} \\ -1 & \text{if } ij \text{ are noncyclic order.} \end{cases} \quad \text{in two dimension}$$

$$e_{ijk} e_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \qquad e_{km} e_{nm} = \delta_{km}$$

$$e_{ijk} A_{jk} = 0 \quad \text{and} \quad e_{jk} A_{jk} = 0 \quad \text{IF } A_{jk} = A_{kj} \tag{8.1}$$

- The product of the permutation function with a function that is symmetric in its subscript will result in a zero value.

Derivative notation

The partial derivative of a function with respect to a coordinate is shown by a comma followed by the index of the direction.

$$\frac{\partial u_i}{\partial x_j} = u_{i,j}$$

Equations of elasticity and thin plate in indicial notation

tensor normal strains = engineering normal strains   tensor shear strains = (engineering shear strains)/2

$$\text{Tensor strain: } \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\text{Generalized Hooke's law: } \varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}]$$

$$\text{Equilibrium equations: } \sigma_{ij,j} + F_i = 0$$

$$\text{Stress vector: } S_i = \sigma_{ij} n_j$$

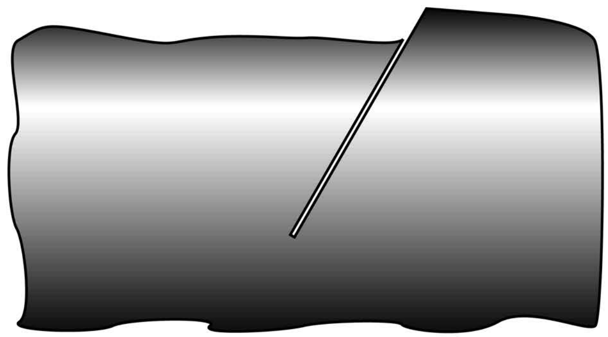
$$\text{Linear strain energy density: } U_0 = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$$

C8.1 Obtain the differential equation governing displacements in a three dimensional elastic body subjected to body forces F_i.

C8.2 Starting with the displacement equation and assuming all assumptions are valid in classical plate theory, obtain equations in indicial notation for strain, stresses, stress resultants, equilibrium equations, differential equation, and strain energy.

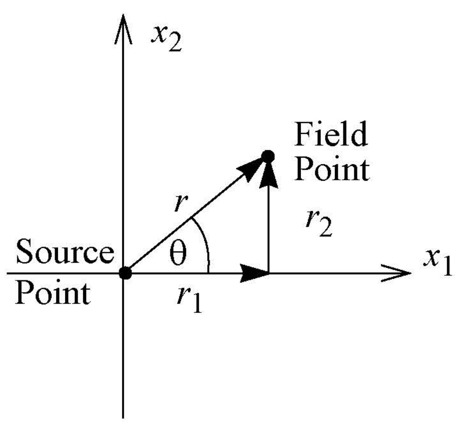
Fundamental solutions

Two adjoining points along a line displace relative to one another in a **dislocation** as shown below.



Intensity of dislocation: D_i

- Point strain singularity called **inclusion singularity.**
- Point stress singularity called **inhomogeneity singularity**.



$$r^2 = r_i r_i \qquad r_{i,j} = \delta_{ij} \qquad r_{,j} = r_j / r \qquad [ln(r)]_{,j} = r_j / r^2$$

$$\theta = atan\left(\frac{r_2}{r_1}\right) \qquad \frac{\partial\theta}{\partial r_1} = -\frac{r_2}{r^2} \qquad \frac{\partial\theta}{\partial r_2} = \frac{r_1}{r^2} \qquad \theta_{,j} = e_{mj}\frac{r_m}{r^2}$$

C8.3 The displacement field due to a dislocation in an elastic body in plane strain is given by the equation below. Obtain the strain invariant ε_{ii}.

$$u_i = \frac{1}{8\pi(1-\nu)}\left[4(1-\nu)\delta_{ik}\theta - 2(1-2\nu)e_{ki}ln(r) + 2e_{kn}\frac{r_i r_n}{r^2}\right]D_k$$

where D_k is the intensity of dislocation.

Variational calculus

- If H is defined in terms of $u_{,i}$ then in taking the derivative $\partial H / \partial u_{,k}$ and we will use $\partial u_{,i} / \partial u_{,k} = \delta_{ik}$.

Stationary value of a functional with first order derivatives

$$I(u) = \iint_A H(u_{,x_1}, u_{,x_2}, u, x_1, x_2) dx_1 dx_2$$

$$\text{Differential Equation: } \frac{\partial H}{\partial u} - \left(\frac{\partial H}{\partial u_{,i}}\right)_{,i} = 0 \qquad x_k \text{ in } A$$

$$\text{Boundary Conditions: } \frac{\partial H}{\partial u_{,i}} n_i = 0 \qquad \text{or} \qquad \delta u_j = 0 \qquad x_k \text{ on } \Gamma$$

Stationary value of a functional with second order derivatives

$$I(u) = \iint_A H(u_{,x_1 x_1}, u_{,x_1 x_2}, u_{,x_2 x_2}, u_{,x_1}, u_{,x_2}, u, x_1, x_2) dx_1 dx_2$$

$$\boldsymbol{m}_{nn} = \left(\frac{\partial H}{\partial u_{,ij}}\right) n_i n_j \qquad \boldsymbol{m}_{nt} = \left(\frac{\partial H}{\partial u_{,ij}}\right) e_{mi} n_m n_j \qquad -\boldsymbol{q}_i = \left(\frac{\partial H}{\partial u_{,i}}\right) - \left(\frac{\partial H}{\partial u_{,ij}}\right)_{,j}$$

$$\text{Differential Equation: } \boldsymbol{q}_{i,i} + \frac{\partial H}{\partial u} = 0 \qquad x_k \text{ in } A$$

$$\text{Boundary Conditions: } \left.\begin{array}{lcl} \boldsymbol{m}_{nn} = 0 & \text{or} & \delta\left(\dfrac{\partial u}{\partial n}\right) = 0 \\ & \text{and} & \\ \dfrac{\partial \boldsymbol{m}_{nt}}{\partial s} + \boldsymbol{q}_i n_i = 0 & \text{or} & \delta u = 0 \end{array}\right\} \quad x_k \text{ on } \Gamma$$

$$\text{Corner Condition: } (\Delta \boldsymbol{m}_{nt})_C = 0 \qquad \text{or} \qquad \delta u_C \qquad x_k \text{ on corner } \Gamma_C$$

C8.4 Obtain the boundary value problem for thin plate bending with only transverse distributed forces using indicial notation and variational calculus.

Nonlinear Strains

A perspective of reference frame

x and y: coordinates of a point in Cartesian system on the *undeformed* body.

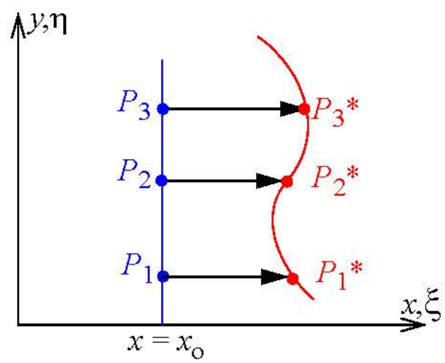
ξ and η: coordinates of a point in Cartesian system on the *deformed* body.

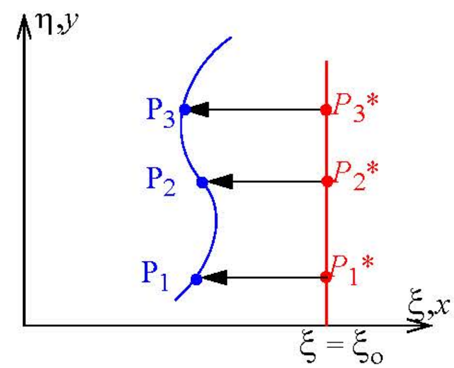
u and v are the displacement components in the x and y direction.

represents a curve in the ξ, η coordinate system if u and v are not linear in y.

$$\xi = x + u(x, y) \qquad \eta = y + \mathrm{v}(x, y) \qquad \xi_o = x + u(x, y) \qquad \eta = y + v(x, y)$$

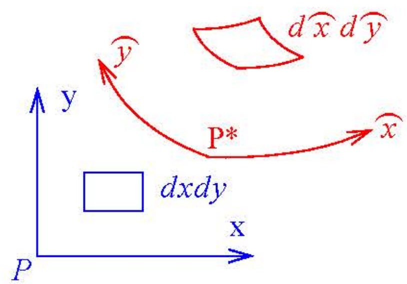
$$\xi = x + u(x_o, y) \qquad \eta = y + \mathrm{v}(x_o, y) \qquad x = f_1(\xi_o, \eta) \qquad y = f_2(\xi_o, \eta)$$

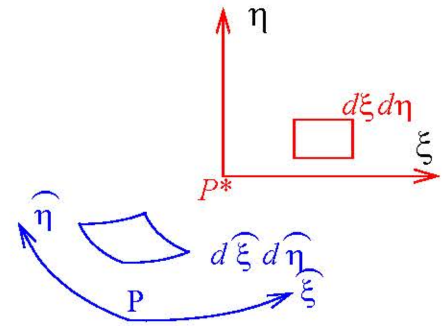




- a curve in the x, y coordinates provided f_1 and f_2 are not a linear function of η.
- straight lines becomes curves in the process of deformation.

- coordinate system of straight edges will become curvilinear coordinate system





- The various measures (definitions) of stress and strain are an outcome of the reference frame we use to describe them.

Non-linear (Finite) strain

Our objective is to determine the change in the distance between two *infinitely* close points of a body as it deforms.

- Lagrangian strain: uses the original undeformed geometry as a reference.
- Eulerian strain: uses the final deformed geometry as a reference.

Lagrangian strain in two dimension

Point $P(x, y)$ deforms to point $P^*(\xi, \eta)$
Point $Q(x+dx, y+dy)$ deforms to $Q^*(\xi+d\xi, \eta+d\eta)$.

$$d\xi = dx + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \left(1 + \frac{\partial u}{\partial x}\right)dx + \frac{\partial u}{\partial y}dy \qquad d\eta = dy + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \frac{\partial v}{\partial x}dx + \left(1 + \frac{\partial v}{\partial y}\right)dy$$

The square of infinitesimal distance PQ and P^*Q^* can be written as:

$$ds^2 = dx^2 + dy^2 \qquad ds^{*2} = d\xi^2 + d\eta^2$$

$$ds^{*2} - ds^2 = 2\varepsilon_{xx}dx^2 + 2\varepsilon_{yy}dy^2 + 2(\varepsilon_{xy} + \varepsilon_{yx})dxdy$$

$$\boxed{\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]}$$

$$\boxed{\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]}$$

$$\boxed{\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial x}\right]}$$

Interpretation

Conventional definition of strain in the direction of PQ.

$$E_{PQ} = \frac{ds^* - ds}{ds} \qquad ds^* = (1 + E_{PQ})ds$$

$$E_{PQ}\left(1 + \frac{1}{2}E_{PQ}\right)ds^2 = \varepsilon_{xx}dx^2 + \varepsilon_{yy}dy^2 + \varepsilon_{xy}dxdy$$

Consider a line element PQ is in the x direction ($E_{PQ} = E_{xx}$).

$ds = dx$, $dy = 0$, and $dz = 0$.

$$E_{xx}\left(1 + \frac{1}{2}E_{xx}\right) = \varepsilon_{xx} \qquad or \qquad E_{xx} = \sqrt{1 + 2\varepsilon_{xx}} - 1$$

Eulerian strain in two dimension

Point $P(x, y)$ deforms to point $P^*(\xi, \eta)$,

Point $Q(x+dx, y+dy)$ deforms to $Q^*(\xi+d\xi, \eta+d\eta)$.

We use the deformed geometry as our reference geometry to describe the problem, that is u and v are now functions of ξ, η.

$$\xi = x + u(\xi, \eta) \qquad \eta = y + v(\xi, \eta)$$
$$x = \xi - u(\xi, \eta) \qquad y = \eta - v(\xi, \eta)$$

We now consider the infinitesimal lengths as:

$$dx = d\xi - \left(\frac{\partial u}{\partial \xi}d\xi + \frac{\partial u}{\partial \eta}d\eta\right) = \left(1 - \frac{\partial u}{\partial \xi}\right)d\xi - \frac{\partial u}{\partial \eta}d\eta$$

$$dy = d\eta - \left(\frac{\partial v}{\partial \xi}d\xi + \frac{\partial v}{\partial \eta}d\eta\right) = -\frac{\partial v}{\partial \xi}d\xi + \left(1 - \frac{\partial v}{\partial \eta}\right)d\eta$$

$$ds^{*2} - ds^2 = 2E_{\xi\xi}d\xi^2 + 2E_{\eta\eta}d\eta^2 + 2(E_{\xi\eta} + E_{\eta\xi})d\xi d\eta$$

$$\boxed{E_{\xi\xi} = \frac{\partial u}{\partial \xi} - \frac{1}{2}\left[\left(\frac{\partial u}{\partial \xi}\right)^2 + \left(\frac{\partial v}{\partial \xi}\right)^2\right]}$$

$$\boxed{E_{\eta\eta} = \frac{\partial v}{\partial \eta} - \frac{1}{2}\left[\left(\frac{\partial v}{\partial \eta}\right)^2 + \left(\frac{\partial u}{\partial \eta}\right)^2\right]}$$

$$\boxed{E_{\xi\eta} = E_{\eta\xi} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \eta}\right) - \frac{\partial u}{\partial \xi}\frac{\partial u}{\partial \eta} - \frac{\partial v}{\partial \eta}\frac{\partial v}{\partial \xi}\right]}$$

Indicial notation and non-linear strain in three dimension

We represent x_i and ξ_i as the coordinates in the undeformed and deformed state and u_i the displacement vector.

x_1, x_2, and x_3 represent x, y, and z, respectively.

ξ_1, ξ_2, and ξ_3 represent ξ, η, and ζ, respectively.

u_1, u_2, and u_3 represent u, v, and w, respectively.

A repeated index implies summation.

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right]}$$

$$\boxed{E_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial \xi_j} + \frac{\partial u_j}{\partial \xi_i} - \frac{\partial u_k}{\partial \xi_i}\frac{\partial u_k}{\partial \xi_j}\right]}$$

- ε_{ij}: Green's strain tensor---Lagrangian description of finite strain in tensor form
- E_{ij}: Almansi's strain tensor---the Eulerian description of finite strain in tensor form. In other words, the is the For small strain the two tensor form are the same
- Both strain tensors are symmetric: $\varepsilon_{ij} = \varepsilon_{ji}$ $\quad E_{ij} = E_{ji}$

C8.5 Lagrangian strain with moderately large rotation for plate bending is given as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}w_{,i}w_{,j} \qquad i,j = x, y$$

Assume the displacement field is $u_i = u_{oi} - zw_{,i}$, where u_{oi} and w are the inplane and bending displacements, respectively. All assumptions of classical plate theory except small strain approximation are valid. p_i and p_z are the inplane and bending distributed loads per unit area. Obtain the stress formulas and the statement of boundary value problems for displacements u_{oi} and w.