

Variational and Energy Methods

The learning objectives in this chapter are:

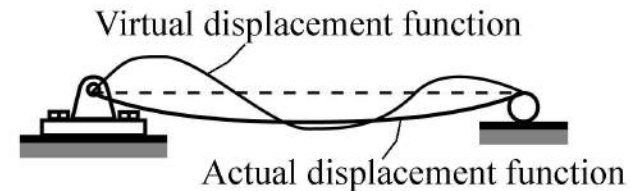
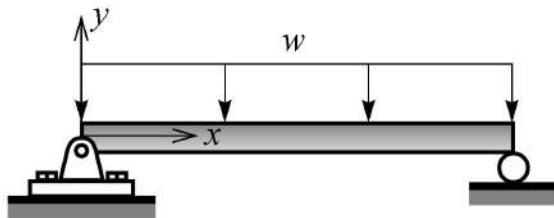
- Understand the concepts in variational calculus.
- Understand the application of variational calculus to obtain boundary value problems in mechanics of materials.
- Understand the use of variational calculus in approximate methods of Rayleigh-Ritz and Finite Element Method.

Basic Concepts

Variational calculus is the branch of mathematics dealing with finding the maximum and minimum values of functionals.

Functionals: Function of a function. Strain energy, the potential of work, and potential energy are all functionals of the displacements which are functions of the position coordinates.

1. We can walk to different points and measure the elevation— we will call it the **d-process**; so dx , ds , du represent the **actual movement** along a path (curve).
2. We can conduct a thought experiment. For example, without moving, we ask the question, if we go to that point will the elevation increase or decrease? This imaginary movement is called the **virtual movement** or the δ -process.



Independent set of functions

$$a_1 u_1 + a_2 u_2 + a_3 u_3 + \dots + a_n u_n = 0$$

If $u_1, u_2, u_3, \dots, u_n$ are independent functions then above equation implies $a_i = 0 \quad i = 1 \text{ to } n$

- Any set of independent variables (parameters) that describes the system geometry are called the **generalized coordinates**.
- The space spanned by the generalized coordinates is called the **configuration space**.
- Any condition that limits the change in geometry in the configuration space is called the **kinematic condition**.
- Functions that are *continuous* and satisfy *all the kinematic boundary conditions* are called **kinematically admissible functions**.

Extremum and Stationary Values

Find minimum of $F = F(u_1, u_2, \dots + u_n)$

We consider a virtual change in the configuration space, that is, space spanned by the independent variables $u_1, u_2, u_3, \dots, u_n$. The total virtual change δF is the sum of the slopes multiplied by virtual change in each direction.

$$\delta F = \frac{\partial F}{\partial u_1} \delta u_1 + \frac{\partial F}{\partial u_2} \delta u_2 + \frac{\partial F}{\partial u_3} \delta u_3 \dots + \frac{\partial F}{\partial u_n} \delta u_n$$

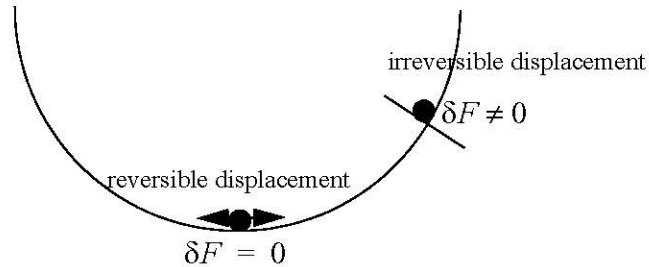
- δF is called the **first variation** of F . If F is to be a minimum at a point in the configuration space, then this change of δF must be zero.

$$\delta F = \frac{\partial F}{\partial u_1} \delta u_1 + \frac{\partial F}{\partial u_2} \delta u_2 + \frac{\partial F}{\partial u_3} \delta u_3 \dots + \frac{\partial F}{\partial u_n} \delta u_n = 0$$

If $u_1, u_2, u_3, \dots, u_n$ are independent variables then we are free to move in any direction. So if we only walk in u_1 (all other virtual displacement are zero) then we have $\partial F / \partial u_1 = 0$. In a similar manner we can walk in each of the directions and conclude:

$$\frac{\partial F}{\partial u_i} = 0 \quad i = 1, 2, 3 \dots n$$

- For stationary values we need the virtual displacement to be **reversible**.



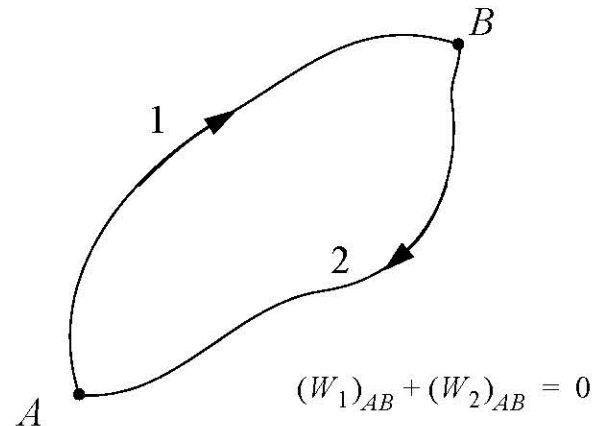
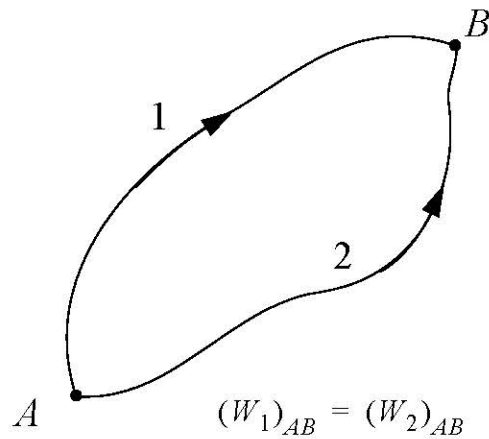
Functionals

- A **function** $u(x)$ is a rule of correspondence such that for all x in D there is assigned a unique element $u(x)$ in R . A **functional** $F[u(x)]$ is a rule of correspondence such that for all $u(x)$ in R there is assigned a unique element $F[u(x)]$ in Ω . In other words, a functional is a function of a function.
- Linear functional:** $l(\alpha_1 u + \alpha_2 v) = \alpha_1 l(u) + \alpha_2 l(v)$
- Bilinear functional** $B(\alpha_1 u_1 + \alpha_2 u_2, v) = \alpha_1 B(u_1, v) + \alpha_2 B(u_2, v)$ $B(u, \alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 B(u, v_1) + \alpha_2 B(u, v_2)$
- Symmetric bilinear functional:** $B(u, v) = B(v, u)$
- u and v can be vectors.

Work

Work is done by a force if the point at which the force is applied moves. If the point at which force \bar{F} is applied moves through an infinitesimal distance $d\bar{u}$, then the work is defined as

$$W = \bar{F} \cdot d\bar{u}$$



- Work done by a force is **conserved** if it is path independent.
- Friction, permanent deformation are examples in which work will not be conserved.
- Rubber has nonlinear stress–strain curve. Work done in stretching rubber is recovered when the forces are released and the rubber returns to the undeformed position.

Nonlinear systems and non-conservative systems are two independent descriptions of a system.

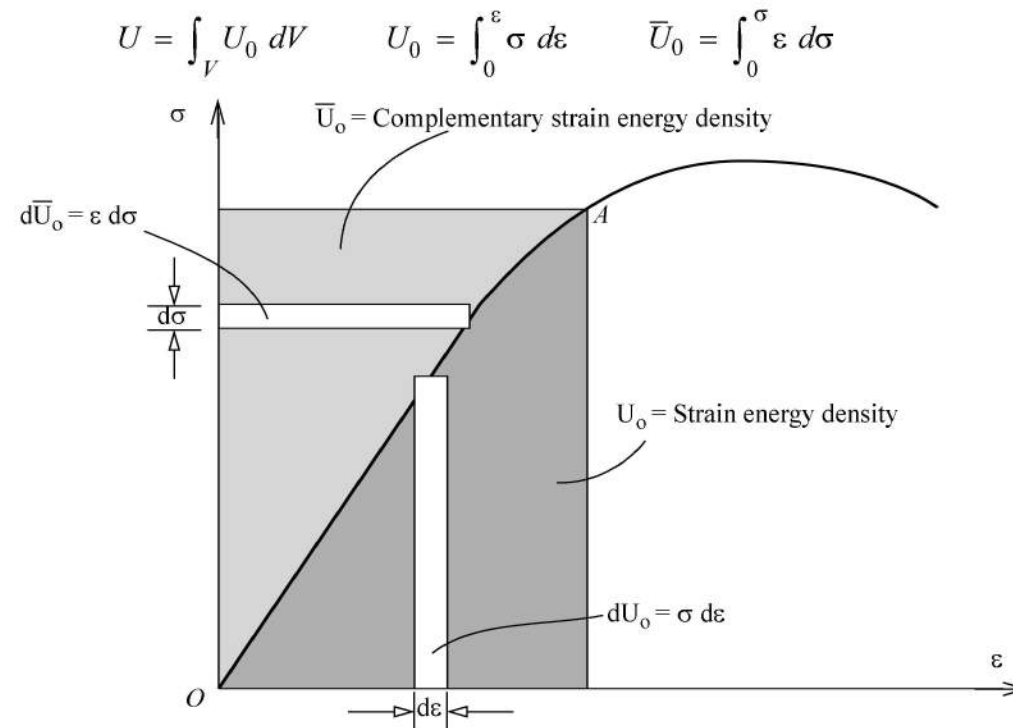
- Work is a scalar quantity. Work from different types of forces and moments can be added.

Table 1: Work Expressions

	Work
Axial	$W_A = \int_L p_x(x)u(x)dx + \sum_{q=1}^m F_q u(x_q) = l(u)$
Torsion of circular shafts	$W_T = \int_L t(x)\phi(x)dx + \sum_{q=1}^m T_q \phi(x_q) = l(\phi)$
Symmetric bending of beams	$W_B = \int_L p_y(x)v(x)dx + \sum_{q=1}^{m_1} F_q v(x_q) + \sum_{q=1}^{m_2} M_q \frac{dv}{dx}(x_q) = l(v)$
Bending of thin plates	$W_P = \iint_A p_z(x,y)w(x,y)dxdy = l(w)$
Plane stress elasticity	$W_E = h \iint_A [F_x(x,y)u(x,y) + F_y(x,y)v(x,y)]dxdy = l(u,v)$

Strain Energy

- The change in internal energy in a body during deformation is called the **strain energy**.
- The energy per unit volume is called the **strain energy density** and is the area under the stress–strain curve up to the point of deformation.



- The units for strain energy density are newton-meters per cubic meter ($\text{N} \cdot \text{m}/\text{m}^3$), joules per cubic meter (J/m^3), inch-pounds per cubic inch ($\text{in} \cdot \text{lb}/\text{in}^3$), and foot-pounds per cubic foot ($\text{ft} \cdot \text{lb}/\text{ft}^3$).

Linear Strain Energy Density

$$U_0 = \frac{1}{2}\sigma\varepsilon \quad U_0 = \frac{1}{2}\tau\gamma$$

- Strain energy, hence strain energy density, is a scalar quantity. We can add the strain energy density due to individual stress and strain components to obtain the total linear strain energy density during deformation.

$$U_0 = \frac{1}{2}[\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}]$$

Linear strain energy in symmetric bending of beams

Two nonzero stress components, σ_{xx} and τ_{xy} . $\sigma_{xx} = E\varepsilon_{xx}$ and $\varepsilon_{xx} = -y(d^2v/dx^2)$

$$U_B = \int_V \frac{1}{2}E\varepsilon_{xx}^2 dV = \int_L \left[\int_A \frac{1}{2}E \left(y \frac{d^2v}{dx^2} \right)^2 dA \right] dx = \int_L \left[\frac{1}{2} \left(\frac{d^2v}{dx^2} \right)^2 \int_A Ey^2 dA \right] dx = \frac{1}{2} \int_L EI_{zz} \left(\frac{d^2v}{dx^2} \right)^2 dx$$

The strain energy due to shear in bending is $U_S = (1/2) \int \tau_{xy}\gamma_{xy} dV$.

The maximum shear stress τ_{xy} and shear strain γ_{xy} are an order of magnitude smaller than the maximum normal stress σ_{xx} and the maximum normal strain ε_{xx} . U_S will be two orders of magnitude smaller than U_B and can be neglected in our calculations.

Linear strain energy in bending of thin plates

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2} \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = -z \frac{\partial^2 w}{\partial y^2} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z \frac{\partial^2 w}{\partial x \partial y}$$

$$\sigma_{xx} = \frac{E}{(1-\nu^2)}(\varepsilon_{xx} + \nu\varepsilon_{yy}) \quad \sigma_{yy} = \frac{E}{(1-\nu^2)}(\varepsilon_{yy} + \nu\varepsilon_{xx}) \quad \tau_{xy} = G\gamma_{xy}$$

Linear strain energy in plane stress elasticity

$$\varepsilon_{xx} = \sigma_{xx} - \nu\sigma_{yy} \quad E\varepsilon_{yy} = \sigma_{yy} - \nu\sigma_{xx} \quad G\gamma_{xy} = \tau_{xy}$$

$$\sigma_{xx} = E[\varepsilon_{xx} + \nu\varepsilon_{yy}]/(1-\nu^2) \quad \sigma_{yy} = E[\varepsilon_{yy} + \nu\varepsilon_{xx}]/(1-\nu^2) \quad \tau_{xy} = G\gamma_{xy}$$

Strain Energy and bilinear functional form of strain energy

Strain Energy	
Axial	$U_A = \frac{1}{2} \int_L EA \left(\frac{du}{dx} \right)^2 dx; \quad U_A = \frac{1}{2} \int_L \left[EA \frac{du_1}{dx} \frac{du_2}{dx} \right] dx = \frac{1}{2} B(u_1, u_2)$
Torsion of circular shafts	$U_T = \frac{1}{2} \int_L GJ \left(\frac{d\phi}{dx} \right)^2 dx; \quad U_T = \frac{1}{2} \int_L \left[GJ \frac{d\phi_1}{dx} \frac{d\phi_2}{dx} \right] dx = \frac{1}{2} B(\phi_1, \phi_2)$
Symmetric bending of beams	$U_B = \frac{1}{2} \int_L EI_{zz} \left(\frac{d^2 v}{dx^2} \right)^2 dx; \quad U_B = \frac{1}{2} \int_L \left[EI_{zz} \frac{d^2 v_1}{dx^2} \frac{d^2 v_2}{dx^2} \right] dx = \frac{1}{2} B(v_1, v_2)$
Thin Plates	$U_P = \frac{D}{2} \iint_A \left\{ \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 2\nu \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial y^2} \right) + 2(1-\nu) \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} dx dy$ $U_P = \frac{D}{2} \iint_A \left[\frac{\partial^2 w_1}{\partial x^2} \frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2} \frac{\partial^2 w_2}{\partial y^2} + \nu \left(\frac{\partial^2 w_1}{\partial x^2} \frac{\partial^2 w_2}{\partial y^2} + \frac{\partial^2 w_2}{\partial x^2} \frac{\partial^2 w_1}{\partial y^2} \right) + 2(1-\nu) \frac{\partial^2 w_1}{\partial x \partial y} \frac{\partial^2 w_2}{\partial x \partial y} \right] dx dy = \frac{1}{2} B(w_1, w_2)$
Plane Stress Elasticity	$U_E = \frac{Eh}{2(1-\nu^2)} \iint_A \left[\left(\frac{\partial u}{\partial x} \right)^2 + 2\nu \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial v}{\partial y} \right) + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \frac{(1-\nu)}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 dx dy$ $U_E = \frac{Eh}{2(1-\nu^2)} \iint_A \left[\frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial x} + \nu \left(\frac{\partial u_2}{\partial x} \frac{\partial v_1}{\partial y} + \frac{\partial u_1}{\partial x} \frac{\partial v_2}{\partial y} \right) + \frac{\partial v_1}{\partial y} \frac{\partial v_2}{\partial y} \right] + \frac{(1-\nu)}{2} \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} \right) \left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x} \right) dx dy$ $= \frac{1}{2} B(u_1, v_1, u_2, v_2)$

Virtual Work

- Virtual work methods are applicable to linear and nonlinear systems, to conservative as well as non-conservative systems.

The total virtual work done on a body at equilibrium is zero.

- Virtual work implies it is not actual work but work done by actual forces in moving points through *virtual displacements*, or, virtual forces moving through actual displacement.

$$\delta W = 0 \quad \delta W_{\text{ext}} = \delta W_{\text{int}}$$

Minimum Potential Energy

- We define the potential energy function as: $\Omega = U - W$
 U is the strain energy and W is the work potential of a force.
- The “work potential of a force” is associated with conservative forces only and implies that there is a potential function from which such a force can be obtained.
- Minimum potential energy and methods derived from it are applicable to conservative systems that can be linear or non-linear.

The internal virtual work is the variation in elastic strain energy during deformation: $\delta W_{\text{int}} = \delta U$

The external virtual work is the variation in the work potential of the force: $\delta W_{\text{ext}} = \delta W$

$$\delta W_{\text{int}} - \delta W_{\text{ext}} = \delta U - \delta W = 0 \text{ or } \delta \Omega = 0$$

The virtual variation in the potential energy function is zero—which occurs where the slopes of the potential energy function with respect to the parameters defining the potential function are zero.

Of all the kinematically admissible displacement functions, the actual displacement function is the one that minimizes the potential energy function at stable equilibrium.

There are many kinematically admissible displacement functions, and there is no requirement that these functions satisfy the equilibrium equations or the boundary conditions on forces and moments.

- The actual displacement is kinematically admissible and satisfies all the equilibrium conditions and the static boundary conditions.
- If we choose an arbitrary kinematically admissible function and calculate the potential energy function, the value so obtained will always be greater than the value of the potential energy function at equilibrium.
- The better approximation of displacement function is the one that yields the lower potential energy.
- The greater the degrees of freedom, the lower will be the potential energy for a given set of kinematically admissible functions.

$$\Omega = \frac{1}{2}B(u, u) - l(u)$$

Mathematical preliminaries

Transfer of derivatives from one function to another inside an integral is done by: *integration by parts* in 1-D; by *Green's formula* in 2-D; and by *Gauss divergence formula* in 3-D.

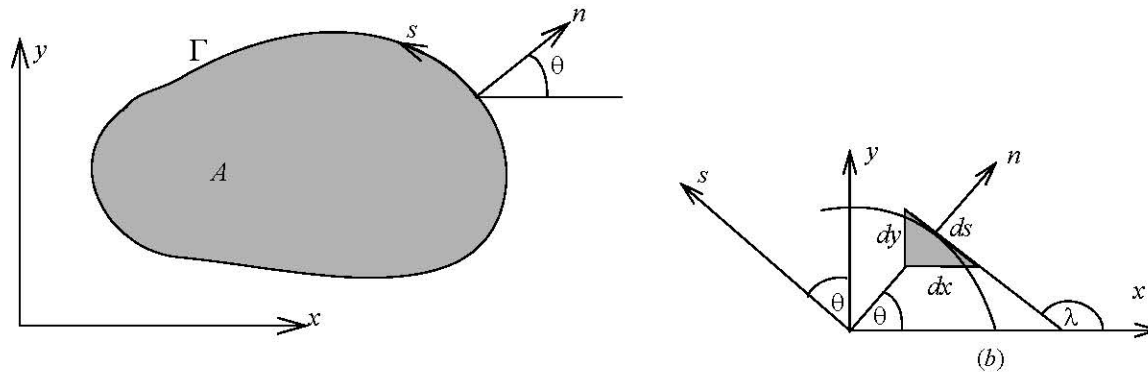
Integration by parts is given by the equation below.

$$\int_a^b f \frac{dg}{dx} dx = fg \Big|_a^b - \int_a^b g \frac{df}{dx} dx$$

The *Green's formulas* is given by the equation below.

$$\iint_A f \frac{\partial g}{\partial x} dx dy = \oint_{\Gamma} fg dy - \iint_A g \frac{\partial f}{\partial x} dx dy \qquad \iint_A f \frac{\partial g}{\partial y} dx dy = - \oint_{\Gamma} fg dx - \iint_A g \frac{\partial f}{\partial y} dx dy$$

where $f(x, y)$ and $g(x, y)$ are two continuous functions in the area A that is bounded by the curve Γ .



We can relate the Cartesian coordinates (x, y) and to normal and tangential coordinates (n, s) .

$$\begin{aligned} x &= n \cos \theta - s \sin \theta & y &= n \sin \theta + s \cos \theta \\ n &= x \cos \theta + y \sin \theta & s &= -x \sin \theta + y \cos \theta \end{aligned}$$

We define the direction cosines of the unit normal as

$$n_x = \frac{\partial n}{\partial x} = \cos \theta = \frac{\partial s}{\partial y} \qquad n_y = \frac{\partial n}{\partial y} = \sin \theta = -\frac{\partial s}{\partial x}$$

If a point is restricted to the boundary, then $dn = 0$

$$dx = (-\sin\theta)ds = -n_y ds \quad dy = (\cos\theta)ds = n_x ds$$

$$\iiint_A f \frac{\partial g}{\partial x} dx dy = \oint_{\Gamma} (fg) n_x ds - \iint_A g \frac{\partial f}{\partial x} dx dy \quad \iiint_A f \frac{\partial g}{\partial y} dx dy = \oint_{\Gamma} (fg) n_y dy - \iint_A g \frac{\partial f}{\partial y} dx dy$$

$$\text{By Chain rule: } \frac{\partial f}{\partial x} = \frac{\partial f}{\partial n} \frac{\partial n}{\partial x} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial x} = \frac{\partial f}{\partial n} n_x - \frac{\partial f}{\partial s} n_y \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial n} \frac{\partial n}{\partial y} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial y} = \frac{\partial f}{\partial n} n_y + \frac{\partial f}{\partial s} n_x$$

$$\frac{\partial n_x}{\partial s} = -\sin\theta \left(\frac{\partial \theta}{\partial s} \right) = -n_y \left(\frac{\partial \theta}{\partial s} \right) = -\left(\frac{n_y}{R_{cur}} \right) \quad \frac{\partial n_y}{\partial s} = \cos\theta \left(\frac{\partial \theta}{\partial s} \right) = n_x \left(\frac{\partial \theta}{\partial s} \right) = \frac{n_x}{R_{cur}}$$

where, R_{cur} is the radius of curvature of the boundary at the point under consideration and can be function of s .

Gauss divergence formula is given by the equation below.

$$\iiint_T \left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right] dV = \iint_S [u_x n_x + u_y n_y + u_z n_z] dA$$

where, u_x , u_y , and u_z are the components of a vector function which is continuous with continuous first derivatives in a region T bounded by a smooth surface S ; and n_x , n_y , and n_z are the direction cosines of a unit normal on surface S . To develop formulas in which we transfer derivatives x , y , or z from one function to another we let each of the components of u be equal to product of two continuous functions fg while the other two components are zero. This produces the formulas below.

$$\begin{aligned} \iiint_T \frac{\partial f}{\partial x} g dV &= \iint_S fg n_x dA - \iiint_T f \frac{\partial g}{\partial x} dV \\ \iiint_T \frac{\partial f}{\partial y} g dV &= \iint_S fg n_y dA - \iiint_T f \frac{\partial g}{\partial y} dV \\ \iiint_T \frac{\partial f}{\partial z} g dV &= \iint_S fg n_z dA - \iiint_T f \frac{\partial g}{\partial z} dV \end{aligned}$$

Stationary Value Of A Definite Line Integral

Notation: $u^{(0)} = u$ $u^{(i)} = du/dx$ $u^{(ii)} = d^2u/dx^2$ • • $u^{(r)} = d^r u/dx^r$

Stationary value of a functional with first order derivatives

$$I(u) = \int_a^b H(u^{(i)}, u, x) dx$$

$$\text{First variation: } \delta I(u) = \delta \int_a^b H(u^{(i)}, u, x) dx = \int_a^b \delta H(u^{(i)}, u, x) dx = \int_a^b \left[\frac{\partial H}{\partial u^{(i)}} \delta u^{(i)} + \frac{\partial H}{\partial u} \delta u \right] dx$$

- Function and derivative are independent in virtual displacement.
- Once we have considered virtual displacement, we are now on a specific curve and function and its derivative are related are no longer independent. We perform integration by parts.

$$\delta I(u) = \int_a^b \left[\frac{\partial H}{\partial u} - \frac{d}{dx} \left(\frac{\partial H}{\partial u^{(i)}} \right) \right] \delta u dx + \frac{\partial H}{\partial u^{(i)}} \delta u \Big|_a^b = 0$$

Possibility 1: We meet the condition $\delta I(u) = 0$ in the average or overall sense. This lead to approximate methods as the condition is not satisfied at each and every point between a and b .

Possibility 2: We require that $\delta I(u) = 0$ at each and every point between a and b . This results in boundary value problem.

- During the process of variation, the function and its derivative are independent.
- After the process of variation, the function and its derivative are no longer independent.
- Integration by parts will generate an expression only in terms of variation of the function.

Boundary Value Problem

$$\text{Differential Equation: } \frac{\partial H}{\partial u} - \frac{d}{dx} \left(\frac{\partial H}{\partial u^{(i)}} \right) = 0 \quad a < x < b$$

$$\text{Boundary Conditions: } \left[\frac{\partial H}{\partial u^{(i)}} = 0 \quad \text{or} \quad \delta u = 0 \right] \quad \text{at } x = a \text{ and at } x = b$$

The above equations are called the *Euler-Lagrange* equations.

Stationary value of a functional with second order derivatives

$$I(u) = \int_a^b H(u^{(ii)}, u^{(i)}, u, x) dx \quad (7.1a)$$

We take the first variation $\delta I(u)$ and then by successive integration by parts we transfer derivatives.

$$\begin{aligned} \delta I(u) &= \int_a^b \left[\frac{\partial H}{\partial u^{(ii)}} \delta u^{(ii)} + \frac{\partial H}{\partial u^{(i)}} \delta u^{(i)} + \frac{\partial H}{\partial u} \delta u \right] dx \\ \delta I(u) &= \frac{\partial H}{\partial u^{(ii)}} \delta u^{(i)} \Big|_a^b + \left\{ \frac{\partial H}{\partial u^{(i)}} - \frac{d}{dx} \left(\frac{\partial H}{\partial u^{(ii)}} \right) \right\} \delta u \Big|_a^b + \int_a^b \left[\frac{\partial H}{\partial u} - \frac{d}{dx} \left(\frac{\partial H}{\partial u^{(i)}} \right) + \frac{d^2}{dx^2} \left(\frac{\partial H}{\partial u^{(ii)}} \right) \right] \delta u dx = 0 \end{aligned}$$

Boundary value problem

Differential Equation: $\frac{\partial H}{\partial u} - \frac{d}{dx} \left(\frac{\partial H}{\partial u^{(i)}} \right) + \frac{d^2}{dx^2} \left(\frac{\partial H}{\partial u^{(ii)}} \right) = 0 \quad a < x < b$

Boundary Conditions:

$$\frac{\partial H}{\partial u^{(ii)}} = 0 \quad \text{or} \quad \delta u^{(i)} = 0$$

and

at $x = a$ and at $x = b$

$$\frac{\partial H}{\partial u^{(i)}} - \frac{d}{dx} \left(\frac{\partial H}{\partial u^{(ii)}} \right) = 0 \quad \text{or} \quad \delta u = 0$$

Generalization: $I(u) = \int_a^b H(u^{(r)}, u^{(r-1)}, u^{(r-1)}, \dots, u^{(1)}, u^{(0)}, x) dx$

Differential equation:
$$\frac{\partial H}{\partial u^{(0)}} - \frac{d}{dx} \left(\frac{\partial H}{\partial u^{(1)}} \right) + \frac{d^2}{dx^2} \left(\frac{\partial H}{\partial u^{(2)}} \right) - \frac{d^3}{dx^3} \left(\frac{\partial H}{\partial u^{(3)}} \right) + \dots + (-1)^r \frac{d^r}{dx^r} \left(\frac{\partial H}{\partial u^{(r)}} \right) = 0$$

Boundary conditions at $x = a$ and $x = b$

$$-\left(\frac{\partial H}{\partial u^{(1)}} \right) + \frac{d}{dx} \left(\frac{\partial H}{\partial u^{(2)}} \right) - \frac{d^2}{dx^2} \left(\frac{\partial H}{\partial u^{(3)}} \right) + \dots + (-1)^r \frac{d^{(r-1)}}{dx^{(r-1)}} \left(\frac{\partial H}{\partial u^{(r)}} \right) = 0 \quad \text{or} \quad \delta u = 0$$

$$\left(\frac{\partial H}{\partial u^{(2)}} \right) - \frac{d}{dx} \left(\frac{\partial H}{\partial u^{(3)}} \right) + \dots + (-1)^r \frac{d^{(r-2)}}{dx^{(r-2)}} \left(\frac{\partial H}{\partial u^{(r)}} \right) = 0 \quad \text{or} \quad \delta u^{(1)} = 0$$

$$\left(\frac{\partial H}{\partial u^{(3)}} \right) + \dots + (-1)^r \frac{d^{(r-3)}}{dx^{(r-3)}} \left(\frac{\partial H}{\partial u^{(r)}} \right) = 0 \quad \text{or} \quad \delta u^{(2)} = 0$$

$$\dots = 0 \quad \text{or} \quad \dots = 0$$

$$\dots = 0 \quad \text{or} \quad \dots = 0$$

$$\dots = 0 \quad \text{or} \quad \dots = 0$$

$$\frac{\partial H}{\partial u^{(r)}} = 0 \quad \text{or} \quad \delta u^{(r-1)} = 0$$

1. If the highest derivative in the functional is r , then the differential equation will be of order $2r$.
2. The boundary conditions with the variation symbol of δ have derivatives from 0 to $r-1$. These quantities must be continuous and are called **primary variables**. The boundary conditions are called **kinematic boundary conditions** or **essential boundary conditions**.
3. The boundary conditions that have derivatives of the functionals that vary from r to $2r-1$ are our internal forces and moments and are called **statical variables** or **secondary variables**. The boundary conditions on these variables are called **statical boundary conditions** or **natural boundary conditions**.
4. If the functional contains more than one variable (u), say u_i , then we could replace u with u_i in the above equations.
5. If the functional is quadratic in u and its derivative, then the boundary value problem will be linear. However, the above equations are applicable to any functional.

C7.1 Obtain the boundary value problem for symmetric bending of beams using variational principles. Assume there are only distributed force p_y and no concentrated force or moment acting on the beam.

Stationary value of a definite area integral

- In stationary value of an area integral the transfer of derivative is accomplished by using Green's theorem.

Stationary value of a functional with first order derivatives

$$I(u) = \iint_A H(u_{,x}, u_{,y}, u, x, y) dx dy \quad \text{where} \quad u_{,x} = \partial u / \partial x \quad u_{,y} = \partial u / \partial y$$

- A subscript without a comma implies a component and with a comma implies derivative.

$$\delta I(u) = \iint_A \left[\left(\frac{\partial H}{\partial u_{,x}} \right) \delta u_x + \left(\frac{\partial H}{\partial u_{,y}} \right) \delta u_y + \left(\frac{\partial H}{\partial u} \right) \delta u \right] dx dy$$

$$\delta I(u) = \oint_{\Gamma} \left[\left(\frac{\partial H}{\partial u_{,x}} \right) n_x + \left(\frac{\partial H}{\partial u_{,y}} \right) n_y \right] \delta u ds + \iint_A \left[\left(\frac{\partial H}{\partial u} \right) - \frac{\partial}{\partial x} \left(\frac{\partial H}{\partial u_{,x}} \right) - \frac{\partial}{\partial y} \left(\frac{\partial H}{\partial u_{,y}} \right) \right] \delta u dx dy = 0$$

$$\text{Differential Equation: } \frac{\partial H}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial H}{\partial u_{,x}} \right) - \frac{\partial}{\partial y} \left(\frac{\partial H}{\partial u_{,y}} \right) = 0 \quad x, y \text{ in } R$$

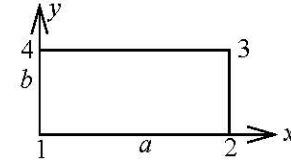
$$\text{Boundary Conditions: } \left(\frac{\partial H}{\partial u_{,x}} \right) n_x + \left(\frac{\partial H}{\partial u_{,y}} \right) n_y = 0 \quad \text{or} \quad \delta u = 0 \quad x, y \text{ on } S$$

C7.2 Obtain the boundary value problem for an elastic plane in plane stress with only body forces.

Stationary value of a functional with second order derivatives in rectangular coordinates

Green's Formula

$$\iint_A f \frac{\partial g}{\partial x} dx dy = \oint_{\Gamma} f g dy - \iint_A g \frac{\partial f}{\partial x} dx dy \quad \iint_A f \frac{\partial g}{\partial y} dx dy = - \oint_{\Gamma} f g dx - \iint_A g \frac{\partial f}{\partial y} dx dy$$



$$I(u) = \iint_A H(u_{,xx}, u_{,xy}, u_{,yy}, u_{,x}, u_{,y}, u, x, y) dx dy \quad \text{where} \quad u_{,xx} = \partial^2 u / \partial x^2 \quad u_{,yy} = \partial^2 u / \partial y^2 \quad u_{,xy} = \partial^2 u / \partial x \partial y$$

$$\delta I(u) = \iint_A \left[\left(\frac{\partial H}{\partial u} \right) \delta u + \left(\frac{\partial H}{\partial u_{,x}} \right) \delta u_{,x} + \left(\frac{\partial H}{\partial u_{,y}} \right) \delta u_{,y} + \left(\frac{\partial H}{\partial u_{,xx}} \right) \delta u_{,xx} + \left(\frac{\partial H}{\partial u_{,yy}} \right) \delta u_{,yy} + \left(\frac{\partial H}{\partial u_{,xy}} \right) \delta u_{,xy} \right] dx dy$$

Differential Equation:
$$\frac{\partial^2}{\partial x^2} \left(\frac{\partial H}{\partial u_{,xx}} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial H}{\partial u_{,yy}} \right) + \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial H}{\partial u_{,xy}} \right) - \frac{\partial}{\partial x} \left(\frac{\partial H}{\partial u_{,x}} \right) - \frac{\partial}{\partial y} \left(\frac{\partial H}{\partial u_{,y}} \right) + \left(\frac{\partial H}{\partial u} \right) = 0$$

Boundary Conditions

$$\frac{\partial H}{\partial u_{,xx}} = 0 \quad \text{or} \quad \delta u_{,x} = 0 \quad \text{at } x = 0 \text{ and } x = a$$

$$\frac{\partial}{\partial x} \left(\frac{\partial H}{\partial u_{,xx}} \right) + \frac{\partial}{\partial y} \left(\frac{\partial H}{\partial u_{,xy}} \right) - \left(\frac{\partial H}{\partial u_{,x}} \right) = 0 \quad \text{or} \quad \delta u = 0$$

$$\frac{\partial H}{\partial u_{,yy}} = 0 \quad \text{or} \quad \delta u_{,y} = 0 \quad \text{at } y = 0 \text{ and } y = b$$

$$\frac{\partial}{\partial y} \left(\frac{\partial H}{\partial u_{,yy}} \right) + \frac{\partial}{\partial x} \left(\frac{\partial H}{\partial u_{,xy}} \right) - \left(\frac{\partial H}{\partial u_{,y}} \right) = 0 \quad \text{or} \quad \delta u = 0$$

Corner Conditions:
$$\left(\frac{\partial H}{\partial u_{,xy}} \right)_k = 0 \quad \text{or} \quad \delta u_k = 0 \quad k = 1 \text{ to } 4$$

C7.3 Obtain the boundary value problem for a thin homogeneous plate bending with only transverse distributed forces and correlate the variables to those introduced in plate theory.

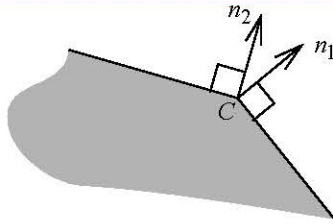
Boundary value problem for geometries with curvilinear boundaries

$$I(u) = \iint_A H(u_{,xx}, u_{,xy}, u_{,yx}, u_{,yy}, u_{,x}, u_{,y}, u, x, y) dx dy$$

$$\delta I(u) = \iint_A \left[\left(\frac{\partial H}{\partial u_{,xx}} \right) \delta u_{,xx} + \left(\frac{\partial H}{\partial u_{,xy}} \right) \delta u_{,xy} + \left(\frac{\partial H}{\partial u_{,yx}} \right) \delta u_{,yx} + \left(\frac{\partial H}{\partial u_{,yy}} \right) \delta u_{,yy} + \left(\frac{\partial H}{\partial u_{,x}} \right) \delta u_{,x} + \left(\frac{\partial H}{\partial u_{,y}} \right) \delta u_{,y} + \left(\frac{\partial H}{\partial u} \right) \delta u \right] dx dy$$

$$\mathbf{m}_{nn} = \left(\frac{\partial H}{\partial u_{,xx}} \right) n_x^2 + \left(\frac{\partial H}{\partial u_{,yy}} \right) n_y^2 + \left(\frac{\partial H}{\partial u_{,xy}} \right) n_x n_y \quad \mathbf{m}_{nt} = - \left(\frac{\partial H}{\partial u_{,xx}} - \frac{\partial H}{\partial u_{,yy}} \right) n_x n_y + \frac{1}{2} (n_x^2 - n_y^2) \frac{\partial H}{\partial u_{,xy}}$$

$$\mathbf{q}_x = \left(\frac{\partial H}{\partial u_{,x}} \right) - \frac{\partial}{\partial x} \left(\frac{\partial H}{\partial u_{,xx}} \right) - \frac{1}{2} \frac{\partial}{\partial y} \left(\frac{\partial H}{\partial u_{,xy}} \right) \quad \mathbf{q}_y = \left(\frac{\partial H}{\partial u_{,y}} \right) - \frac{1}{2} \frac{\partial}{\partial x} \left(\frac{\partial H}{\partial u_{,xy}} \right) - \frac{\partial}{\partial y} \left(\frac{\partial H}{\partial u_{,yy}} \right) \quad \mathbf{q}_n = \mathbf{q}_x n_x + \mathbf{q}_y n_y \quad V_n = \mathbf{q}_n + \frac{\partial \mathbf{m}_{nt}}{\partial s}$$



We represent the discontinuity in \mathbf{m}_{nt} at each corner as $(\Delta \mathbf{m}_{nt})_C$

$$\text{Differential equation: } \frac{\partial \mathbf{q}_x}{\partial x} + \frac{\partial \mathbf{q}_y}{\partial y} + \left(\frac{\partial H}{\partial u} \right) = 0 \quad x, y \text{ in } A$$

Boundary Conditions:

$$\begin{aligned} \mathbf{m}_{nn} &= 0 & \text{or} & & \delta \left(\frac{\partial u}{\partial n} \right) &= 0 \\ \text{and} & & & & & x, y \text{ on } \Gamma \\ V_n &= 0 & \text{or} & & \delta u &= 0 \end{aligned}$$

Corner Condition:

$$(\Delta \mathbf{m}_{nt})_C = 0 \quad \text{or} \quad \delta u_C \quad x, y \text{ on corner } \Gamma_C$$

$$\text{Evaluation of } V_n: V_n = \mathbf{q}_n - \frac{\partial}{\partial s} \left(\frac{\partial H}{\partial u_{,xx}} - \frac{\partial H}{\partial u_{,yy}} \right) n_x n_y + \frac{\partial}{\partial s} \left(\frac{\partial H}{\partial u_{,xy}} \right) \frac{(n_x^2 - n_y^2)}{2} - \left(\frac{\partial H}{\partial u_{,xx}} - \frac{\partial H}{\partial u_{,yy}} \right) \frac{n_x^2 - n_y^2}{R_{cur}} - \left(\frac{\partial H}{\partial u_{,xy}} \right) \frac{2n_x n_y}{R_{cur}}$$

- If the functional contains more than one variable then all the above equations must be written for each variable.

C7.4 Obtain the boundary value problem for thin plate bending with only transverse distributed forces and correlate the variables to those introduced in plate theory.

Stationary value of a definite volume integral

$$I(u) = \iiint_T H(u_{,x}, u_{,y}, u_{,z}, u, x, y, z) dx dy dz \quad \text{where} \quad u_{,x} = \partial u / \partial x \quad u_{,y} = \partial u / \partial y \quad u_{,z} = \partial u / \partial z$$

$$\delta I(u) = \iiint_T \left[\left(\frac{\partial H}{\partial u_{,x}} \right) \delta u_{,x} + \left(\frac{\partial H}{\partial u_{,y}} \right) \delta u_{,y} + \left(\frac{\partial H}{\partial u_{,z}} \right) \delta u_{,z} + \left(\frac{\partial H}{\partial u} \right) \delta u \right] dx dy dz$$

$$\delta I(u) = \iint_S \left[\frac{\partial H}{\partial u_{,x}} n_x + \frac{\partial H}{\partial u_{,y}} n_y + \frac{\partial H}{\partial u_{,z}} n_z \right] \delta u dS + \iiint_T \left[\frac{\partial H}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial H}{\partial u_{,x}} \right) - \left(\frac{\partial}{\partial y} \left(\frac{\partial H}{\partial u_{,y}} \right) - \frac{\partial}{\partial z} \left(\frac{\partial H}{\partial u_{,z}} \right) \right) \right] \delta u dx dy dz = 0$$

Boundary value problem

Differential Equation: $\frac{\partial H}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial H}{\partial u_{,x}} \right) - \frac{\partial}{\partial y} \left(\frac{\partial H}{\partial u_{,y}} \right) - \frac{\partial}{\partial z} \left(\frac{\partial H}{\partial u_{,z}} \right) = 0 \quad x, y, z \text{ in } T$

Boundary Conditions: $\left(\frac{\partial H}{\partial u_{,x}} \right) n_x + \left(\frac{\partial H}{\partial u_{,y}} \right) n_y + \left(\frac{\partial H}{\partial u_{,z}} \right) n_z = 0 \quad \text{or} \quad \delta u = 0 \quad x, y, z \text{ on } S$

- If the functional contains more than one variable then all the above equations must be written for each variable.

Rayleigh-Ritz Method

- Rayleigh-Ritz method is applicable to conservative systems that may be linear or non-linear systems.
- Rayleigh-Ritz method is a formal process of minimizing the potential energy using a series of kinematically admissible displacement functions to produce a set of algebraic equations in the unknown constants of the series approximation.

$$u(x) = \sum_{j=1}^n C_j f_j$$

where, f_j are a set of kinematically admissible functions and C_j are constants to be determined.

- C_j are the generalized coordinates as the variation of them represents the variation of the displacement curve

$$\Omega = \frac{1}{2}B(u_1, u_2) - l(u_1) \quad u_1(x) = \sum_{j=1}^n C_j f_j(x) \quad \text{and} \quad u_2(x) = \sum_{k=1}^n C_k f_k(x)$$

$$B(u_1, u_2) = \sum_{j=1}^n \sum_{k=1}^n C_j C_k B(f_j, f_k) \quad l(u_1) = l\left(\sum_{j=1}^n C_j f_j\right) = \sum_{j=1}^n C_j l(f_j)$$

$$\Omega = \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n C_j C_k B(f_j, f_k) - \sum_{j=1}^n C_j l(f_j)$$

We take the first variation of potential energy and set it equal to zero to minimize the potential energy.

$$\delta \Omega = \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n [\delta C_j C_k B(f_j, f_k) + C_j \delta C_k B(f_j, f_k)] - \sum_{j=1}^n \delta C_j l(f_j) = \sum_{j=1}^n \sum_{k=1}^n \delta C_j [B(f_j, f_k) C_k - l(f_j)] = 0$$

$$\sum_{k=1}^n B(f_j, f_k) C_k - l(f_j) = 0 \quad j = 1 \text{ to } n$$

$$\text{Matrix Form: } [K] \{C\} = \{R\} \quad \text{where} \quad K_{jk} = B(f_j, f_k) \quad R_j = l(f_j)$$

$[K]$ is called the stiffness matrix and because the bilinear functional is symmetric, the stiffness matrix is symmetric.

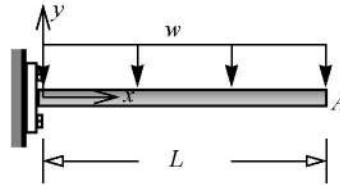
Potential Energy at equilibrium: Let C_j^* represent the solution of the algebraic equations (the values at equilibrium), that is

$$\sum_{k=1}^n B(f_j, f_k) C_k^* = l(f_j)$$

$$\Omega^* = -\frac{1}{2} \sum_{j=1}^n C_j^* l(f_j) = -\frac{1}{2} \sum_{j=1}^n C_j^* R_j = -\frac{1}{2} \{C^*\}^T \{R\} = -W^*/2$$

- At equilibrium, the potential energy of the system is negative of half the work potential.

C7.5 A beam and its loading are as shown below. Use the Rayleigh-Ritz method with one and two parameters to determine the deflection at $x = 0.25L$, $x = 0.5L$, $x = 0.75L$, and $x = L$, and the potential energy function. Compare your results with the analytical solution. Assume that EI is constant for the beam.

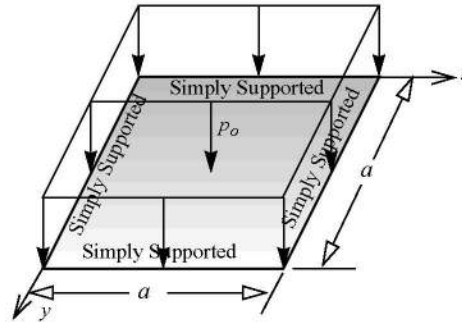


C7.6 A square plate is uniformly loaded and is simply supported on all sides as shown. Using Rayleigh-Ritz method determine deflection w at the center of the plate and the potential energy for each case. (Example 6.5).

Case I: $w(x, y) = C_1 \sin \pi(x/a) \sin \pi(y/a)$

Case II: $w(x, y) = C_1 \sin \pi(x/a) \sin \pi(y/a) + C_2 \sin \pi(2x/a) \sin \pi(y/a) + C_3 \sin \pi(x/a) \sin \pi(2y/a)$

Case III: $w(x, y) = C_1 \sin \pi(x/a) \sin \pi(y/a) + C_2 \sin \pi(3x/a) \sin \pi(y/a) + C_3 \sin \pi(x/a) \sin \pi(3y/a)$



Nonlinearities

- Geometric nonlinearities arise from large deformation.
- Material nonlinearity arises from the nonlinear relationships between stresses and strains
- Contact nonlinearities arise when the contact region between two surfaces change due to applied loads—it requires modifying variational equations and will not be considered any further.

Geometric nonlinearity

- Lagrangian strain is computed from deformation by using the original *undeformed geometry* as the reference geometry.
- Eulerian strain is computed from deformation by using the *final deformed geometry* as the reference geometry.

Material nonlinearity

The only material nonlinearity we will consider is the one in which stress-strain relationship is given by the power law model given below

$$\sigma = \begin{cases} E\varepsilon^n & \varepsilon \geq 0 \\ -E(-\varepsilon)^n & \varepsilon < 0 \end{cases} \quad \text{and} \quad \tau = \begin{cases} G\gamma^n & \gamma \geq 0 \\ -G(-\gamma)^n & \gamma < 0 \end{cases}$$

The strain energy density given below

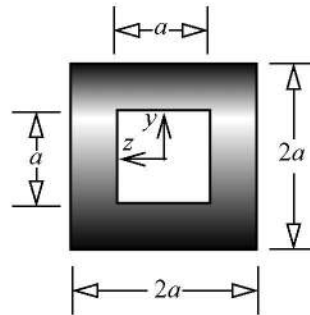
$$U_0 = \begin{cases} E\varepsilon^{n+1}/(n+1) & \varepsilon \geq 0 \\ E(-\varepsilon)^{n+1}/(n+1) & \varepsilon < 0 \end{cases} \quad U_0 = \begin{cases} G\gamma^{n+1}/(n+1) & \gamma \geq 0 \\ G(-\gamma)^{n+1}/(n+1) & \gamma < 0 \end{cases}$$

C7.7 Lagrangian strain with moderately large rotation for symmetric beam bending is given as

$$\varepsilon_{xx} = \frac{du}{dx} + \frac{1}{2} \left(\frac{dv}{dx} \right)^2$$

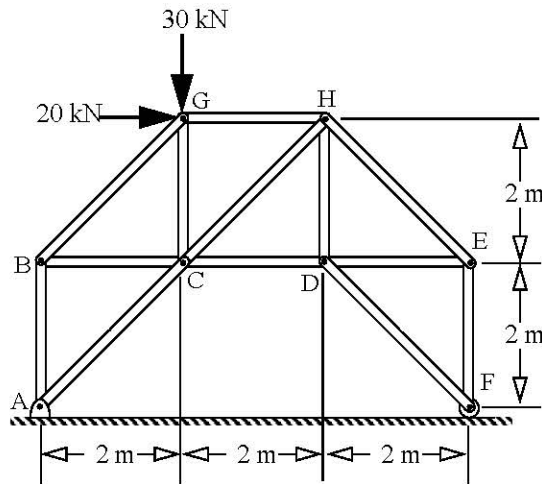
Assume the displacement field is $u(x) = u_o - ydv/dx$, where u_o and v are the axial and bending displacements that are function of x only. All assumptions of classical beam theory except small strain approximation are valid. $p_x(x)$ and $p_y(x)$ are the distributed loads in x and y directions. Obtain the stress formula and the statement of boundary value problems for displacements u_o and v .

C7.8 The hollow square cross section of a beam is shown below. The stress-strain equation of beam material is given by $\sigma = E\epsilon^{0.5}$. The beam is subjected to a transverse distributed force $p_y(x)$. Obtain the boundary value problem for the beam deflection v if all assumptions of classical beam are valid except for the Hooke's law.



Finite Element Method (FEM)

- Our objective is to show how Rayleigh-Ritz method can be used in formulation of one of the versions of the finite element method. Two versions: the **stiffness method** and the **flexibility method**.



Stiffness method: Displacement of pins as unknowns, write the equilibrium of forces at each joint to obtain algebraic equations. ---Potential Energy

Flexibility method: Internal forces in members are unknowns, write compatibility equations at each joint to obtain algebraic equations.---- complimentary potential energy.

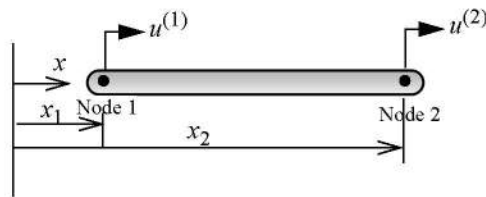
- Rayleigh-Ritz method--- kinematically admissible displacement functions at the global level.
- FEM --- kinematically admissible displacement functions defined piece wise continuous over small (finite) domains called **elements**.
- The kinematically admissible functions are called **interpolation functions**.
- The constants multiplying the piece wise kinematically admissible functions are the displacements of the **nodes**.
- The representation of a structure by elements and nodes is called a **mesh**.
- A mesh with boundary conditions, applied loads, and material property is called a **model**.

Potential energy is a scalar quantity and can be written as the sum of the potential energy of all the individual structural members $\delta\Omega^{(i)}$ as shown below.

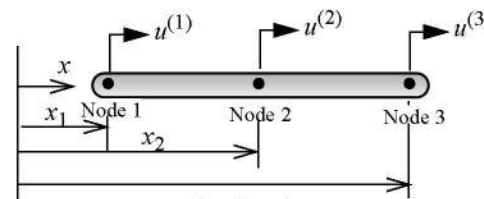
$$\Omega = \sum_{i=1}^n \delta\Omega^{(i)} \quad n \text{ number of elements} \quad (6.2)$$

- We develop matrices at the element level in a local coordinate system without regard to how a member is used in the structure. The individual local matrices are called **element stiffness matrices**.
- The element stiffness matrix are assembled to form the **global stiffness matrix** of the entire structure.
- This perspective of reducing the *complexity of analyzing large structures to the analysis of simple individual members (elements)* is what makes the finite element method such a versatile and popular tool in structural (and engineering application) analysis.
- **Lagrange polynomials** ensure continuity of the function (displacements) at the nodes.
- **Hermite polynomials** ensure continuity of the function and its derivatives at the nodes.
- When the continuity is ensured at the nodes and the boundary then the element is called a **conforming element**.
- When continuity is only ensured at the nodes but not ensured across the boundary then the element is called **non-conforming element**.

Lagrange polynomials in one dimension



linear element



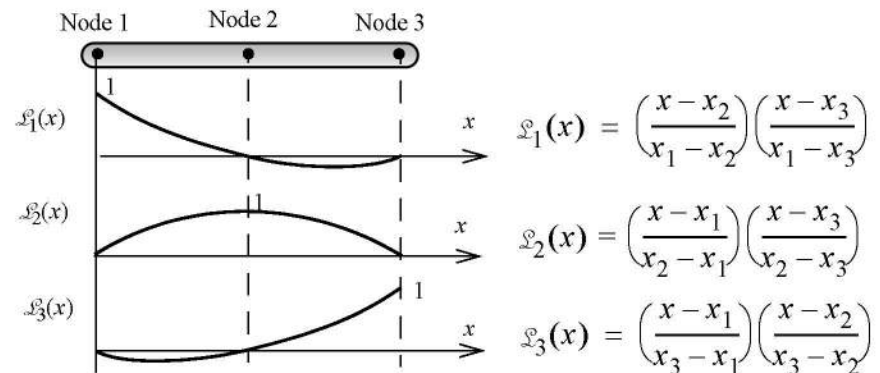
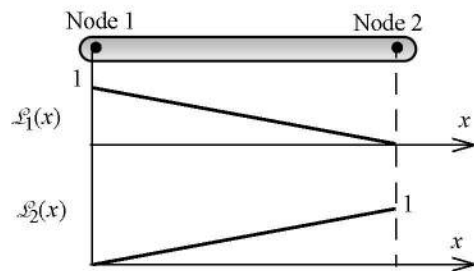
quadratic element

Linear Element: $u(x) = C_1 + C_2x$ $u(x_1) = C_1 + C_2x_1 = u^{(1)}$ $u(x_2) = C_1 + C_2x_2 = u^{(2)}$

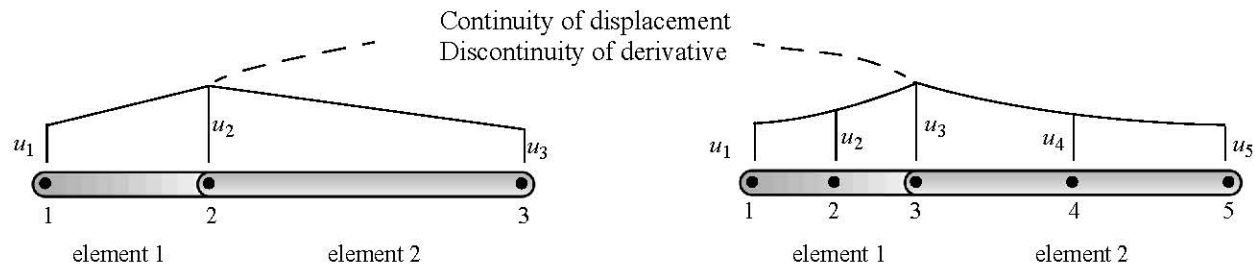
$$u(x) = u^{(1)} \left(\frac{x - x_2}{x_1 - x_2} \right) + u^{(2)} \left(\frac{x - x_1}{x_2 - x_1} \right) = u^{(1)} \mathcal{L}_1(x) + u^{(2)} \mathcal{L}_2(x) = \sum_{i=1}^2 u^{(i)} \mathcal{L}_i(x) \quad \mathcal{L}_1(x) = \left(\frac{x - x_2}{x_1 - x_2} \right) \quad \mathcal{L}_2(x) = \left(\frac{x - x_1}{x_2 - x_1} \right)$$

Quadratic Element: $u(x) = C_1 + C_2x + C_3x^2$ $u(x) = \sum_{i=1}^3 u^{(i)} \mathcal{L}_i(x)$

$$u(x_j) = \sum_{i=1}^n u^{(i)} \mathcal{L}_i(x_j) = u^{(j)} \quad \mathcal{L}_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$



$$\mathcal{L}_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^n \left[\frac{(x - x_j)}{(x_i - x_j)} \right]$$



Natural Coordinates

- Non-dimensional coordinates are called **natural coordinates** in finite element method.

$$\xi = x/L \quad 0 \leq \xi \leq 1 \quad \text{or} \quad \xi = 2x/L \quad -1 \leq \xi \leq 1$$

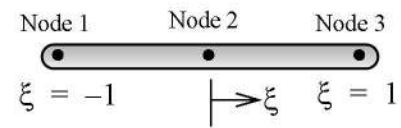
- Natural coordinates used for approximating the geometry as well as the generalized displacements.

$$\text{Curved Beam: } x = \sum_{i=1}^n x_i \mathcal{L}_i(\xi) \quad y = \sum_{i=1}^n y_i \mathcal{L}_i(\xi)$$

- Isoparametric elements** are elements in which geometric transformation and the generalized displacements are approximated by the same interpolation functions.
- Natural coordinates used in numerical integration.

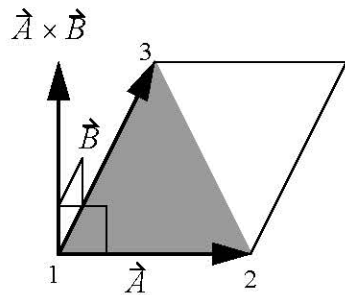
$$\begin{array}{ll} \text{Linear} & \mathcal{L}_1(\xi) = 1 - \xi \quad \mathcal{L}_2(\xi) = \xi \\ \text{Quadratic} & \mathcal{L}_1(\xi) = (1 - \xi)(1 - 2\xi) \quad \mathcal{L}_2(\xi) = 4(1 - \xi)\xi \quad \mathcal{L}_3(\xi) = -\xi(1 - 2\xi) \end{array}$$

C7.9 Obtain the quadratic Lagrange polynomial using the natural coordinate shown below.



Vector arithmetic

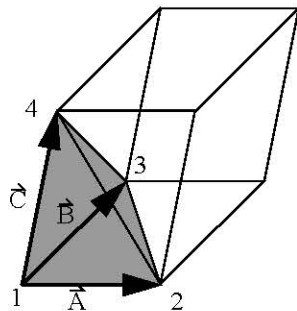
- Vector arithmetic provides a simple way of obtaining formulas for areas of triangles and volumes of tetrahedrons in terms of coordinates of points which we will use for Lagrange polynomials in two and three dimensions.



The cross product $\vec{A} \times \vec{B}$ results in a vector whose magnitude is the area of the parallelogram and direction is perpendicular to the plane formed by the two vectors. The area of the triangle is half of the area of parallelogram formed by the two vectors.

$$\vec{A} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} \quad \vec{B} = (x_3 - x_1)\vec{i} + (y_3 - y_1)\vec{j}$$

$$\text{Area of triangle 1-2-3} = \vec{A} \times \vec{B} / 2 = [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)] / 2$$



The scalar triple product $(\vec{A} \times \vec{B}) \cdot \vec{C}$ results in a quantity whose magnitude is the volume of the parallelepiped and the volume of the tetrahedron is one-sixth of the parallelepiped. The order of multiplication in cross product is in counter-clockwise direction with respect to the positive z-direction.

$$\vec{A} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k} \quad \vec{B} = (x_3 - x_1)\vec{i} + (y_3 - y_1)\vec{j} + (z_3 - z_1)\vec{k}$$

$$\vec{C} = (x_4 - x_1)\vec{i} + (y_4 - y_1)\vec{j} + (z_4 - z_1)\vec{k}$$

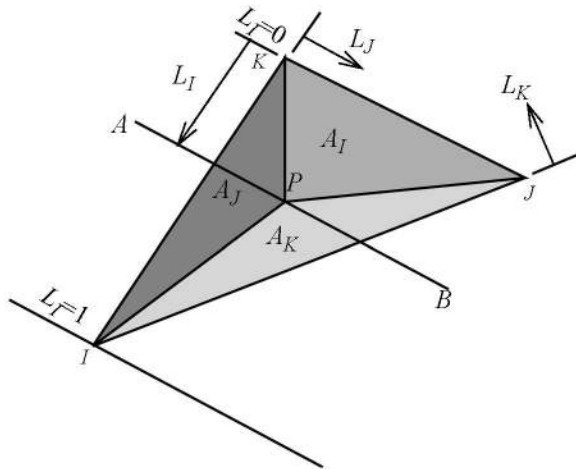
$$\text{Volume of tetrahedron 1-2-3-4} = \frac{1}{6}(\vec{A} \times \vec{B}) \cdot \vec{C} = \frac{1}{6} \begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) \\ (x_4 - x_1) & (y_4 - y_1) & (z_4 - z_1) \end{vmatrix}$$

Lagrange polynomials in two dimensions

Linear: $u(x) = C_0 + C_1x + C_2y$

The triangular element is the simplest element that can be used for modeling regions with curved boundary.

- **Area coordinates** are natural coordinates used in triangular element.



$$L_I = \frac{A_I}{A} \quad L_J = \frac{A_J}{A} \quad L_K = \frac{A_K}{A} \quad L_I + L_J + L_K = 1$$

$$|\overline{PJ} \otimes \overline{PK}| = 2(\text{Area of triangle } PJK)$$

$$A_I = [(x_J - x)(y_K - y) - (x_K - x)(y_J - y)]/2$$

$$A_J = [(x_K - x)(y_I - y) - (x_I - x)(y_K - y)]/2$$

$$A_K = [(x_I - x)(y_J - y) - (x_J - x)(y_I - y)]/2$$

$$A = [(x_J - x_I)(y_K - y_I) - (x_K - x_I)(y_J - y_I)]/2$$

$$\frac{\partial L_I}{\partial x} = \frac{y_{JK}}{2A}$$

$$\frac{\partial L_I}{\partial y} = \frac{-x_{JK}}{2A}$$

where $x_{JK} = x_J - x_K$ and $y_{JK} = y_J - y_K$

$$\frac{\partial L_J}{\partial x} = \frac{y_{KI}}{2A}$$

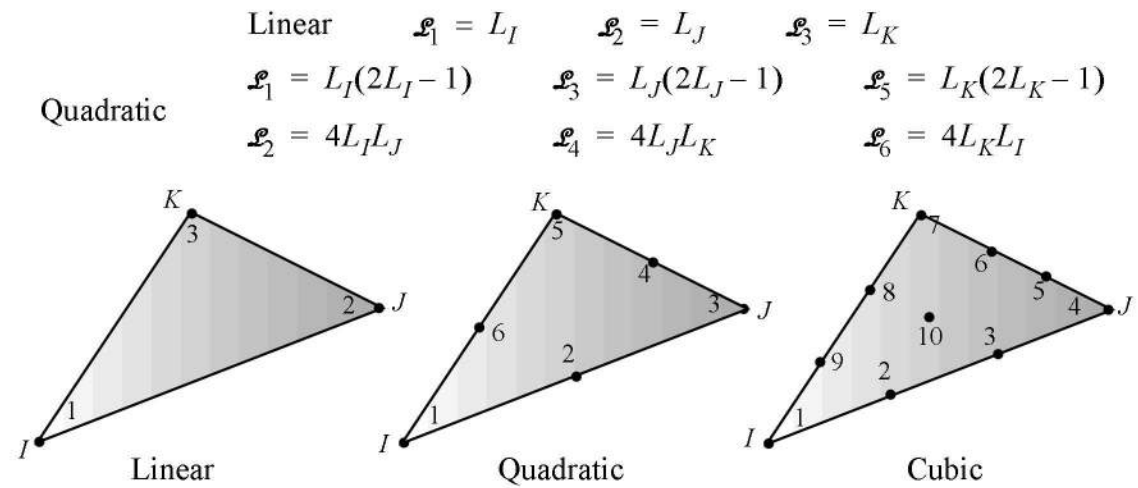
$$\frac{\partial L_J}{\partial y} = \frac{-x_{KI}}{2A}$$

where $x_{KI} = x_K - x_I$ and $y_{KI} = y_K - y_I$

$$\frac{\partial L_K}{\partial x} = \frac{y_{IJ}}{2A}$$

$$\frac{\partial L_K}{\partial y} = \frac{-x_{IJ}}{2A}$$

where $x_{IJ} = x_I - x_J$ and $y_{IJ} = y_I - y_J$



$$\iint_A L_I^m L_J^n L_K^p dx dy = (2A) \frac{m!n!p!}{(m+n+p+2)!}$$

$$\int_a^b L_I^m L_J^n L_K^p ds = (b-a) \frac{m!n!p!}{(m+n+p+1)!}$$

C7.10 Obtain the cubic Lagrange interpolation function for triangular elements for nodes 7, 8, and 9.

Element stiffness matrix and right hand side vector

Method 1: $K_{jk} = B(f_j, f_k) \quad R_j = l(f_j).$

Method II: More than one degree of freedom at each node.

Element displacement vector: $u^{(e)}\}^T = \{u^{(1)}, u^{(2)}, \dots, u^{(n)}\}^T$

Element potential energy $\Omega^{(e)} = U^{(e)} - W^{(e)} \quad U^{(e)} = \frac{1}{2} \{u^{(e)}\}^T [K^{(e)}] \{u^{(e)}\} \quad W^{(e)} = \{R^{(e)}\}^T \{u^{(e)}\}$

$$\delta U^{(e)} = \frac{1}{2} [\{\delta u^{(e)}\}^T [K^{(e)}] \{u^{(e)}\} + \{u^{(e)}\}^T [K^{(e)}] \{\delta u^{(e)}\}] = \frac{1}{2} \{u^{(e)}\}^T ([K^{(e)}] + [K^{(e)}]^T) \{\delta u^{(e)}\} = \{u^{(e)}\}^T [K^{(e)}] \{\delta u^{(e)}\}$$

$$\delta^2 U^{(e)} = \{\delta u^{(e)}\}^T [K^{(e)}] \{\delta u^{(e)}\} = \sum_{i=1}^n \sum_{j=1}^n \delta u^{(i)} K_{ij} \delta u^{(j)} = \frac{\partial^2 U^{(e)}}{\partial u^{(i)} \partial u^{(j)}} \delta u^{(i)} \delta u^{(j)}$$

$$K_{ij} = \frac{\partial^2 U^{(e)}}{\partial u^{(i)} \partial u^{(j)}} \quad R_i = \frac{\partial W^{(e)}}{\partial u^{(i)}}$$

C7.11 Obtain the element stiffness matrix and the element right hand side vector for an axial member using quadratic Lagrange polynomials. Approximate the axial rigidity EA and distributed load p_x with constant values that exist at the center of the axial rod. Also assume the left and right end nodes have concentrated axial forces $F^{(1)}$ and $F^{(3)}$, respectively.

C7.12 An elastic plane of thickness h is in plane stress and subjected to body forces $F_x(x, y)$ and $F_y(x, y)$. For a triangular element, using linear Lagrange polynomial, obtain the first row of the element stiffness matrix and the right hand side vector. Approximate the body forces, the material constant, and thickness with constant values that exists at the centroid of the triangle.(Example 6.7)