Introduction To Elasticity

- Elasticity studies mechanics variables (displacements, strains, stresses, internal forces, and moments) variation with location of point on an elastic body.
- In mechanics of materials the variables are approximated across the cross section or thickness (plates and shells) and integrated cross the cross section or thickness.

The learning objectives in this chapter are:

- Become familiar with equations of elasticity and the Airy stress function.
- Understand the application of equations of elasticity to rotating disks and torsion of non-circular bars.

Basic equations of elasticity

1. Strain-displacement relationship.
2. Constitutive equation: Generalized Hooke's Law.
3. Equilibrium equations on stresses.
4. Compatibility equations on strains to ensure single-valued displacements.
5. Boundary Conditions

Strain-Displacement

Cartesian Coordinates

The strain displacement relationship are given by

$$\boxed{\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}}$$

$$\boxed{\gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \qquad \gamma_{zx} = \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}}$$

Polar coordinates

u_r and v_θ are the displacements in the r and θ direction, respectively

$$\boxed{\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} \qquad \gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}}$$

Compatibility equations

- There are six compatibility equations which ensure that the displacement field obtained from strain fields are single valued.

$$\boxed{\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}}$$

C6.1 Starting with $\dfrac{\partial^2 \varepsilon_{zz}}{\partial x \partial y}$ derive the expression below.

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{1}{2}\frac{\partial}{\partial z}\left[\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right]$$

Constitutive Model: Generalized Hooke's Law

$$\boxed{\begin{aligned} \varepsilon_{xx} &= [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]/E \qquad & \gamma_{xy} &= \tau_{xy}/G \\ \varepsilon_{yy} &= [\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})]/E \qquad & \gamma_{yz} &= \tau_{yz}/G \qquad G = \frac{E}{2(1+\nu)} \\ \varepsilon_{zz} &= [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]/E \qquad & \gamma_{zx} &= \tau_{zx}/G \end{aligned}}$$

plane stress

$$\boxed{E\varepsilon_{xx} = \sigma_{xx} - \nu\sigma_{yy} \qquad E\varepsilon_{yy} = \sigma_{yy} - \nu\sigma_{xx} \qquad G\gamma_{xy} = \tau_{xy}}$$

$$\boxed{\sigma_{xx} = E[\varepsilon_{xx} + \nu\varepsilon_{yy}]/(1-\nu^2) \qquad \sigma_{yy} = E[\varepsilon_{yy} + \nu\varepsilon_{xx}]/(1-\nu^2) \qquad \tau_{xy} = G\gamma_{xy}}$$

plane strain

$$\boxed{2G\varepsilon_{xx} = (1-\nu)\sigma_{xx} - \nu\sigma_{yy} \qquad 2G\varepsilon_{yy} = (1-\nu)\sigma_{yy} - \nu\sigma_{xx} \qquad G\gamma_{xy} = \tau_{xy}}$$

$$\boxed{\sigma_{xx} = \frac{2G}{(1-2\nu)}[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}] \qquad \sigma_{yy} = \frac{2G}{(1-2\nu)}[(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{xx}] \qquad \tau_{xy} = G\gamma_{xy}}$$

$$\boxed{\text{Plane Strain} \qquad \begin{matrix} \nu \to \nu/(1+\nu) \\ G \to G \end{matrix} \qquad \text{Plane Stress}}$$

Alternative:

$$\begin{aligned} 8G\varepsilon_{xx} &= (\kappa+1)\sigma_{xx} - (3-\kappa)\sigma_{yy} \\ 8G\varepsilon_{yy} &= (\kappa+1)\sigma_{yy} - (3-\kappa)\sigma_{xx} \end{aligned}$$

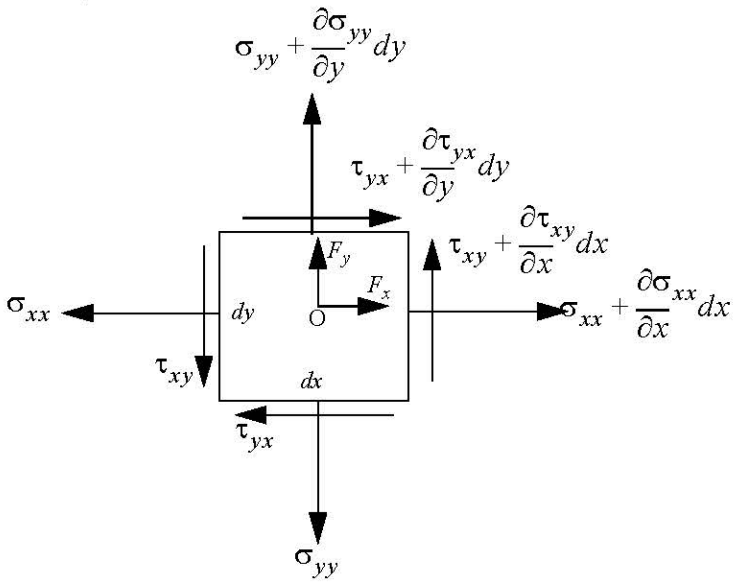
where,

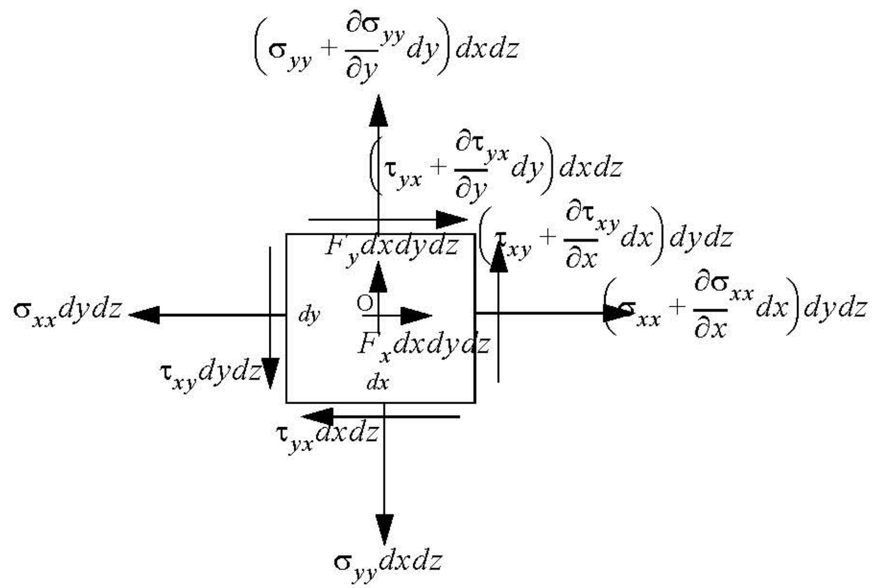
$$\kappa = \begin{cases} \dfrac{(3-\nu)}{(1+\nu)} & \text{Plane Stress} \\ 3 - 4\nu & \text{Plane Strain} \end{cases}$$

Equilibrium equations

Plane Stress Cartesian Coordinates

F_x and F_y are the body forces acting at the point and have the dimensions of *force per unit volume*.





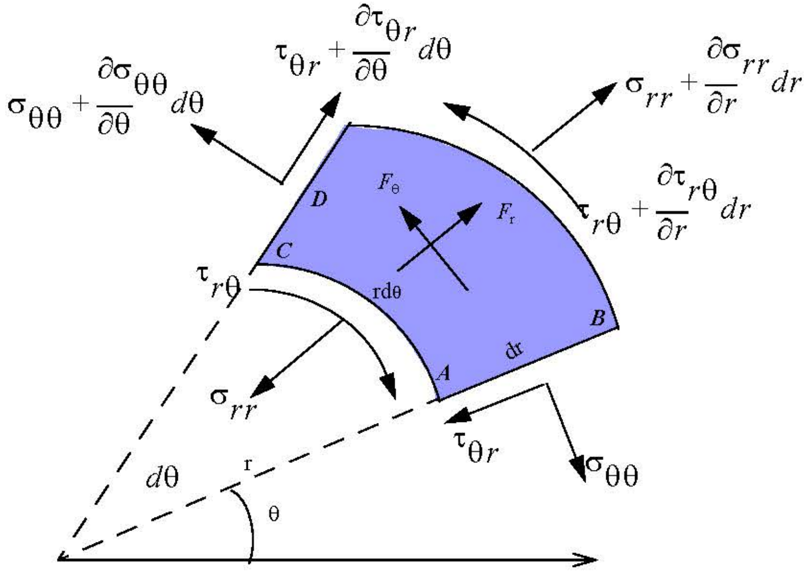
$$\boxed{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = 0} \qquad \boxed{\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0} \qquad \boxed{\tau_{xy} = \tau_{yx}}$$

Three Dimension

$$\boxed{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0} \qquad \boxed{\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0} \qquad \boxed{\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = 0} \qquad \boxed{\sum_j \frac{\partial \sigma_{ji}}{\partial x_j} + F_i = 0}$$

$$\boxed{\tau_{xy} = \tau_{yx}} \qquad \boxed{\tau_{yz} = \tau_{zy}} \qquad \boxed{\tau_{zx} = \tau_{xz}}$$

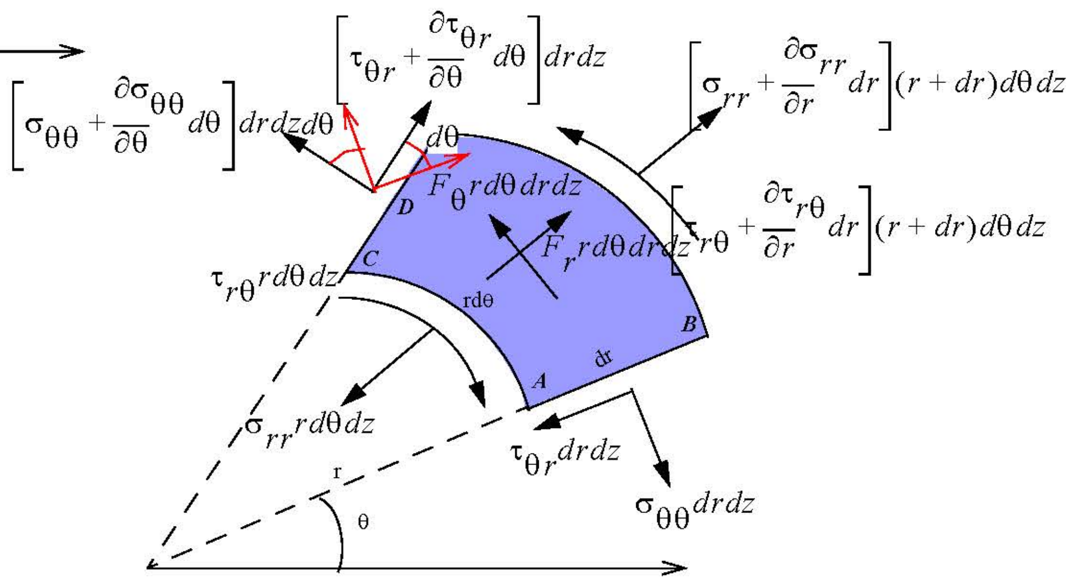
Plane Stress Polar Coordinates



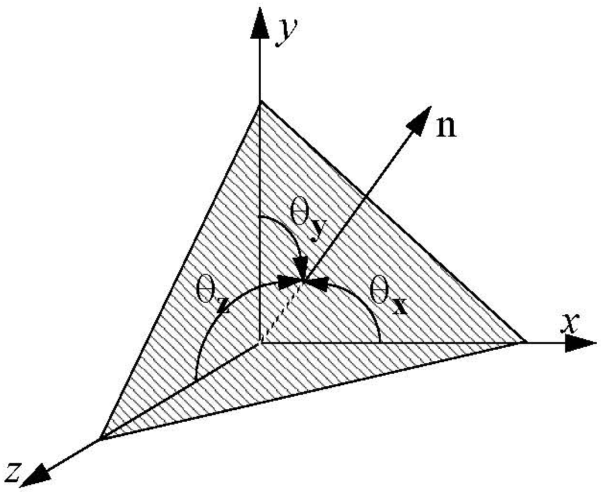
$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r = 0$$

$$\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0$$

$$\tau_{r\theta} = \tau_{\theta r}$$



Boundary conditions



$$\{n\} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \qquad \{S\} = \begin{Bmatrix} S_x \\ S_y \\ S_z \end{Bmatrix} \qquad [\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\{S\} = [\sigma]\{n\}$$

n_x, n_y, and n_z -- direction cosines of the unit normal to the surface in the x, y, and z direction.
S_x, S_y, and S_z -- specified traction in the x, y, and z direction.
u_o,v_o, and w_o -- specified displacement in the x, y, and z direction.

$$\boxed{u = u_o \qquad or \qquad \sigma_{xx}n_x + \tau_{xy}n_y + \tau_{xz}n_z = S_x}$$

$$\boxed{\mathrm{v} = \mathrm{v}_o \qquad or \qquad \tau_{yx}n_x + \sigma_{yy}n_y + \tau_{yz}n_z = S_y}$$

$$\boxed{w = w_o \qquad or \qquad \tau_{zx}n_x + \tau_{zy}n_y + \sigma_{zz}n_z = S_z}$$

C6.2 Starting from the equilibrium equations of elasticity obtain the equilibrium equations of mechanics of materials for axial members and symmetric bending of beams. Assume area of cross-section is a constant.

Axisymmetric problems

- Axisymmetric problems are those in which the loading, geometry, and material properties are all independent of angular location (θ).
- the displacements, strains and stresses should also be independent of θ.

Polar Coordinates: $\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$ $\quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$ $\quad \gamma_{x\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r = 0$$

Axisymmetric: $\boxed{\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \frac{du_r}{dr} \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r}}$

$$\boxed{\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r = 0}$$

Plane Stress Axisymmetric problems

$$\sigma_{rr} = \frac{E}{(1-\nu^2)}[\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}] = \frac{E}{(1-\nu^2)}\left[\frac{du_r}{dr} + \nu\frac{u_r}{r}\right]$$

$$\sigma_{\theta\theta} = \frac{E}{(1-\nu^2)}[\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}] = \frac{E}{(1-\nu^2)}\left[\frac{u_r}{r} + \nu\frac{du_r}{dr}\right]$$

$$\frac{d^2 u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} + \frac{(1-\nu^2)}{E}F_r = 0 \qquad or$$

$$\boxed{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru_r)\right] + \frac{(1-\nu^2)}{E}F_r = 0}$$

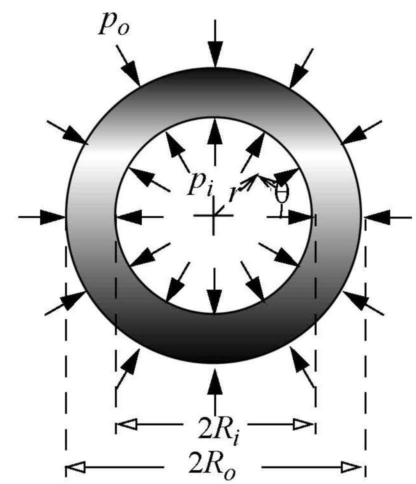
General Solution: $\boxed{u_r = (u_r)_h + (u_r)_p \qquad (u_r)_h = C_1 r + \frac{C_2}{r}}$

C6.3 A thick cylinder (plane strain) is subjected to internal and external pressure as shown. (a) Show that the stresses and radial displacements are given by the equations below. (b) Plot the stresses as a function of r for Case I: $p_o = 0$ and Case II $p_i = 0$

$$\sigma_{rr} = \frac{1}{R_o^2 - R_i^2}\left[-(p_o R_o^2 - p_i R_i^2) - \frac{R_i^2 R_o^2 (p_i - p_o)}{r^2}\right]$$

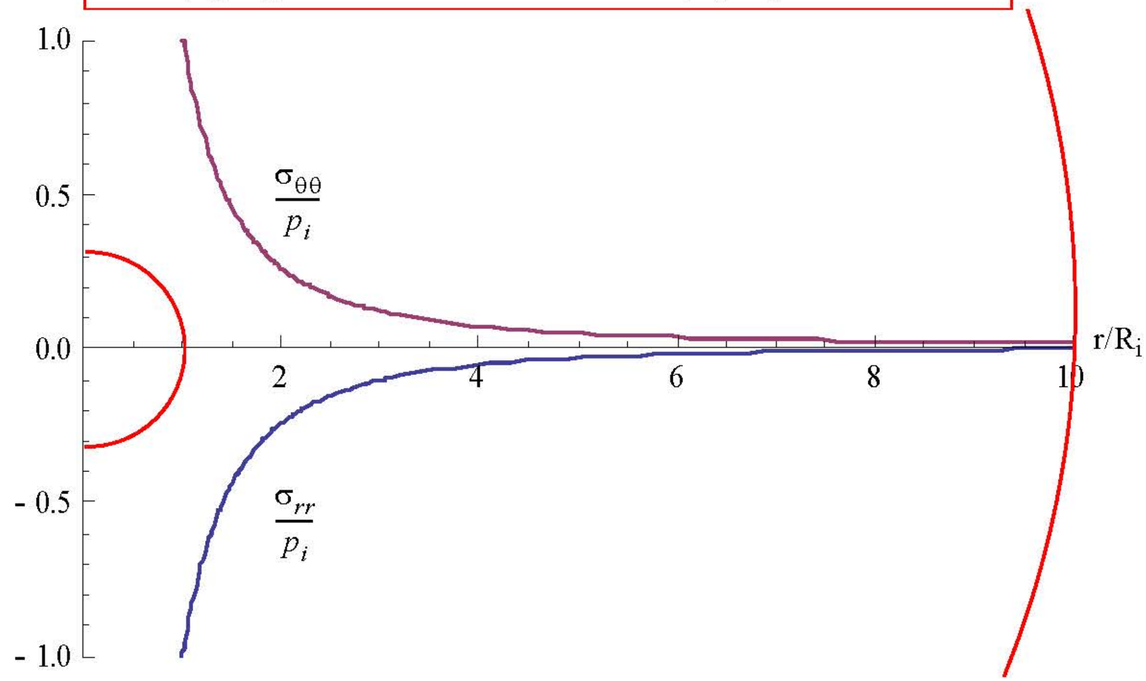
$$\sigma_{\theta\theta} = \frac{1}{R_o^2 - R_i^2}\left[-(p_o R_o^2 - p_i R_i^2) + \frac{R_i^2 R_o^2 (p_i - p_o)}{r^2}\right]$$

$$u_r = \frac{\left[-(1-2\nu)(p_o R_o^2 - p_i R_i^2) r + \dfrac{R_i^2 R_o^2 (p_i - p_o)}{r}\right]}{2G(R_o^2 - R_i^2)}$$



Internal pressure only: $p_o = 0$

$$\sigma_{rr} = \frac{p_i}{(R_o/R_i)^2 - 1}\left[1 - \left(\frac{R_o}{r}\right)^2\right] \qquad \sigma_{\theta\theta} = \frac{p_i}{(R_o/R_i)^2 - 1}\left[1 + \left(\frac{R_o}{r}\right)^2\right]$$



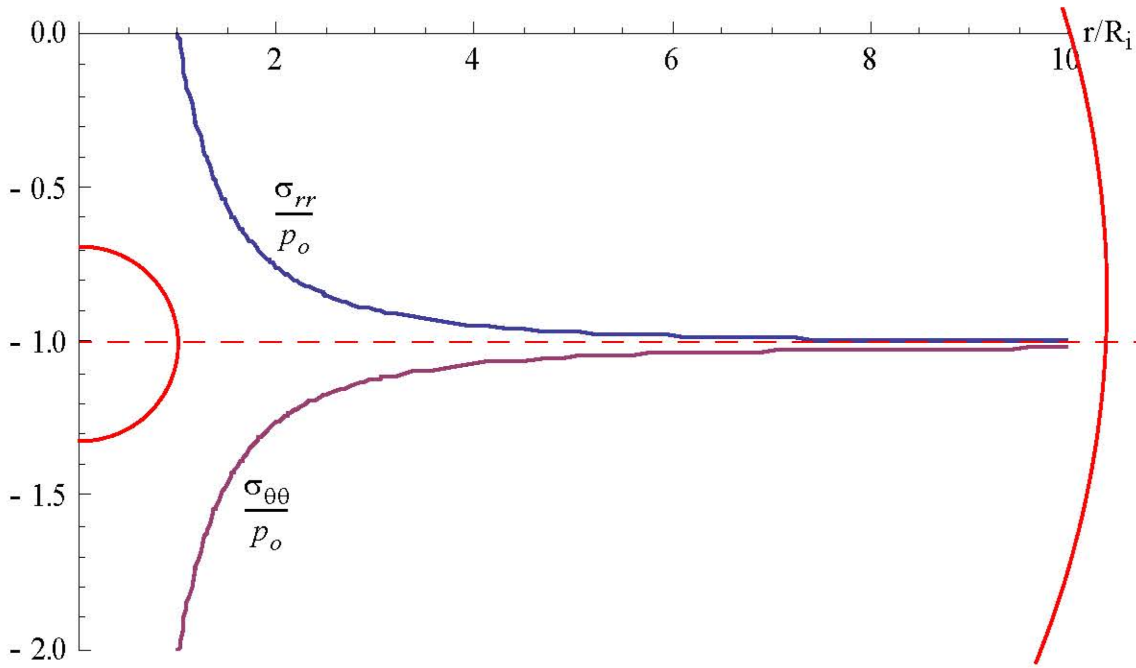
- σ_{rr} is compressive at all points with a maximum compressive value of p_i at the inner surface.
- $\sigma_{\theta\theta}$ is always tensile and its maximum value is also at the inner surface.

$$u_r = \frac{p_i r}{2G[(R_o/R_i)^2 - 1]}\left[(1 - 2\nu) + \left(\frac{R_o}{r}\right)^2\right]$$

- a circle of radius r will enlarge to a radius of $(r + u_r)$ due to the internal pressure.

External pressure only $p_i = 0$

$$\sigma_{rr} = -\left[\frac{p_o}{1-(R_i/R_0)^2}\right]\left[1-\frac{R_i^2}{r^2}\right] \qquad \sigma_{\theta\theta} = -\left[\frac{p_o}{1-(R_i/R_0)^2}\right]\left[1+\frac{R_i^2}{r^2}\right]$$



- σ_{rr} is compressive at all points with a maximum compressive value of p_o at the outer surface.
- $\sigma_{\theta\theta}$ is always compressive and its maximum value is at the inner surface.
- a circle of radius r will shrink to a radius of $(r-u_r)$ due to the external pressure.

$$u_r = -\left[\frac{p_o r}{2G[1-(R_i/R_0)^2]}\right]\left[(1-2\nu)+\left(\frac{R_i}{r}\right)^2\right]$$

C6.4 A small steel cylinder with no axial forces has an inside diameter of 100 mm and an outside diameter of 300 mm. The steel has a modulus of elasticity of E = 200 GPa, Poisson's ratio of 0.3, and a yield stress of σ_{yield} = 200 MPa. Determine the maximum internal pressure if Von Mises stress is not to exceed yield stress.

Rotating disk

- A grinding wheel or a disk brake can be modeled as a rotating disk.

A thin (plane stress) disk is rotating at a constant angular speed ω. $F_r = \rho\omega^2 r$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru_r)\right] + \frac{(1-\nu^2)}{E}(\rho\omega^2 r) = 0$$

$$(u_r)_{part} = -\frac{(1-\nu^2)(\rho\omega^2)}{E}\frac{(r^3)}{8}$$

Displacement

$$\boxed{u_r = C_1 r + \frac{C_2}{r} - \frac{(1-\nu^2)(\rho\omega^2 r^3)}{8E}}$$

Stresses:

$$\boxed{\sigma_{rr} = \frac{E}{(1-\nu^2)}\left[C_1(1+\nu) - \frac{C_2(1-\nu)}{r^2}\right] - \frac{(3+\nu)}{8}\rho\omega^2 r^2}$$

$$\boxed{\sigma_{\theta\theta} = \frac{E}{(1-\nu^2)}\left[C_1(1+\nu) + \frac{C_2(1-\nu)}{r^2}\right] - \frac{(1+3\nu)}{8}\rho\omega^2 r^2}$$

Boundary Conditions

1. A solid rotating disk: The outer boundary is stress free, thus $\sigma_{rr}(r = R_o) = 0$. For solution to be finite at the center (r = 0) of a solid disk requires $C_2 = 0$.
2. A rotating disk with a hole: The inner boundary ($r = R_i$) and outer boundary ($r = R_o$) are stress free. Thus the boundary conditions are $\sigma_{rr}(r = R_i) = 0$ and $\sigma_{rr}(r = R_o) = 0$. (
3. A rotating disk bonded on a rigid shaft: The outer boundary is stress free and the point on in the inner boundary cannot displace. Thus the boundary conditions are $u_r(r = R_i) = 0$ and $\sigma_{rr}(r = R_o) = 0$.

C6.5 The maximum rotational speed at which a grinding wheel can operate is called the “bursting speed,” since if this speed is exceeded, maximum tensile stress will cause the wheel to burst. Consider a grinding wheel with inner radius a, outer radius $2a$, Poisson ratio $\nu = 1/3$, and modulus of elasticity E. Obtain a relationship between the burst speed ω_{max} and the maximum allowable tensile stress σ_{allow} in terms r, a, and E. Assume that the grinding wheel is mounted on a rigid shaft.

Airy Stress Function

- Airy stress function is chosen such that the equilibrium equations in absence of body forces are implicitly satisfied by the stresses in two-dimension.

$$\sigma_{xx} = \frac{\partial^2 \psi}{\partial y^2} \qquad \sigma_{yy} = \frac{\partial^2 \psi}{\partial x^2} \qquad \tau_{xy} = -\left(\frac{\partial^2 \psi}{\partial x \partial y}\right)$$

Equilibrium equations: $\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$

Compatibility equation: $\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$

$$\boxed{\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \qquad \nabla^4 \psi = 0}$$

- Harmonic operator: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
- Bi-harmonic operator $\nabla^4 = \nabla^2 \nabla^2$
- Applicable to plane stress and plane strain.

Solution by polynomials

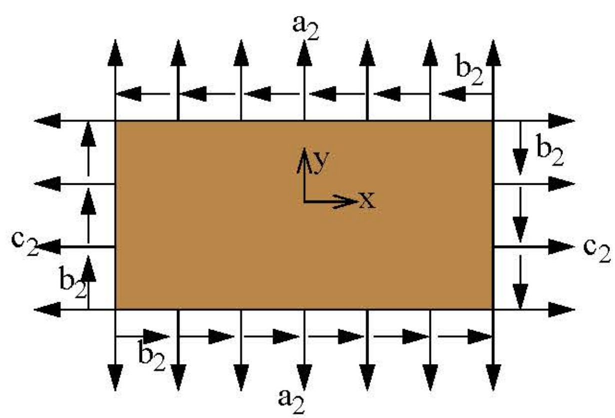
- Airy stress function ψ is represented by a polynomial.
- For stresses to be non-zero the lowest order of polynomial is quadratic.
- Quadratic and cubic polynomials implicitly satisfy the bi-harmonic operator.
- Relationship between constants must be found for polynomials of 4th and higher order for the polynomial to satisfy the bi-harmonic operator.

Quadratic polynomial $\psi = a_2\frac{x^2}{2} + b_2 xy + c_2\frac{y^2}{2}$

Stresses: $\sigma_{xx} = c_2 \qquad \sigma_{yy} = a_2 \qquad \tau_{xy} = -b_2$

- Constant stress state irrespective of the shape of the body.

Tractions: $S_x = c_2 n_x - b_2 n_y \qquad S_y = -b_2 n_x + a_2 n_y$



- If $b_2 = a_2 = 0$, and $c_2 = \sigma$, then we have uni-axial tension.
- If $b_2 = \tau$, and $a_2 = c_2 = 0$, then we have state of pure shear.
- If $b_2 = 0$, and $a_2 = c_2 = \sigma$, then we have the hydrostatic state of stress, i.e., normal stress in all directions is the same σ.
- If $b_2 = 0$, and $a_2 = -c_2 = \sigma$, then we have state of pure shear in a coordinate system that is 45° to the x and y coordinate system.

Displacement from Strains

The strain displacement relationship in two-dimension are given by

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial \mathrm{v}}{\partial y} \qquad \gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial \mathrm{v}}{\partial x}$$

The procedure is as follows:

1. Integrate the strain ε_{xx} with respect to x and add function $f(y)$ to obtain displacement $u(x, y)$.
2. Integrate the strain ε_{yy} with respect to y and add function $g(x)$ to obtain displacement $\mathrm{v}(x, y)$.
3. Substitute $u(x, y)$ and $\mathrm{v}(x, y)$ into the strain expression γ_{xy}. Write all terms that are functions of x on one side of the equal sign and all terms that are functions of y on the other side of the equal sign. This implies that each side must equal to the same constant. Integrate to find $f(y)$ and $g(x)$.
4. The ordinary derivatives of $f(y)$ and $g(x)$ in the above step can be integrated. The constants of integration correspond to rigid body mode and are determined from the boundary conditions.

C6.6 Obtain the displacement field for the constant stress state given below.

$$\sigma_{xx} = c_2 \qquad \sigma_{yy} = a_2 \qquad \tau_{xy} = -b_2$$

Rigid body motion

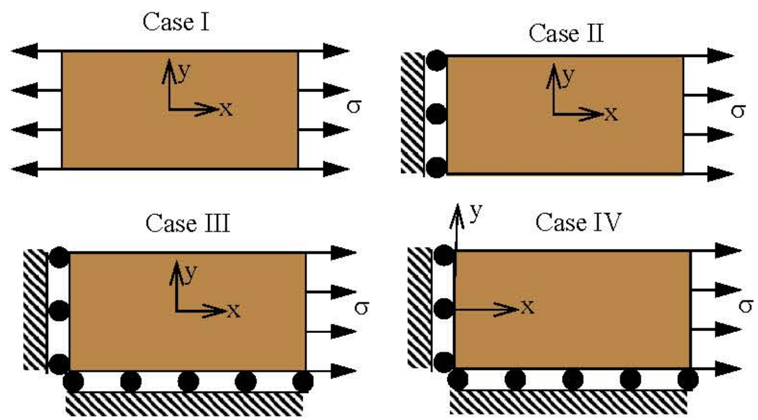
Displacements

$$u(x,y) = \left[\frac{(\kappa+1)}{8G}c_2 - \frac{(3-\kappa)}{8G}a_2\right]x + \alpha_x + \beta y$$

$$\mathrm{v}(x,y) = \left[\frac{(\kappa+1)}{8G}a_2 - \frac{(3-\kappa)}{8G}c_2\right]y - \frac{b_2}{G}x + \alpha_y - \beta x$$

- α_x and α_y represents translation in the x and y direction, respectively.
- the constant β represents rigid body rotation.
- To eliminate rigid body motion we need to fix the body at least at 3 points, with one point that is not co-linear with the other two.

C6.7 Consider the uniaxial tension $\sigma_{xx} = \sigma$ in plane stress and obtain the displacements for the four cases shown. The rectangle is 2a units long in the x direction and 2b units longs in the y direction.



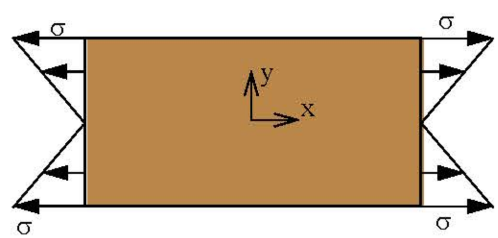
Cubic polynomial $\psi = \frac{a_3}{6}x^3 + \frac{b_3}{2}x^2y + \frac{c_3}{2}xy^2 + d_3\frac{y^3}{6}$

Stresses: $\sigma_{xx} = (c_3x + d_3y)$ $\sigma_{yy} = (a_3x + b_3y)$ $\tau_{xy} = -(b_3x + c_3y)$

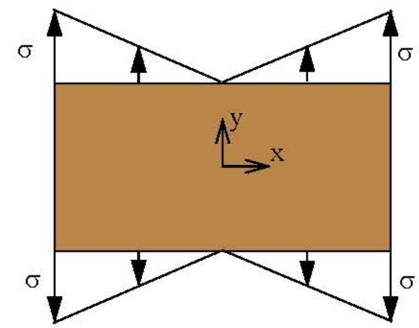
Tractions: $S_x = (c_3x + d_3y)n_x - (b_3x + c_3y)n_y$ $S_y = (b_3x + c_3y)n_x + (c_3x + d_3y)n_y$

- Stress components are linear in x and y irrespective of the shape of the body.
- If $a_3 = b_3 = c_3 = 0$, and $d_3 = \sigma$ then it represents pure bending of a rectangular cross-section as shown in Figure (*a*)
- If $b_3 = c_3 = a_3 = 0$, and $i_3 = \sigma$ then it represents pure bending of a rectangular cross-section as shown in Figure (*b*).

(*a*)



(*b*)



Fourth order polynomial $\psi = \frac{a_4}{12}x^4 + \frac{b_4}{6}x^3y + \frac{c_4}{2}x^2y^2 + \frac{d_4}{6}xy^3 + \frac{e_4}{12}y^4$

$$a_4 + 2c_4 + e_4 = 0$$

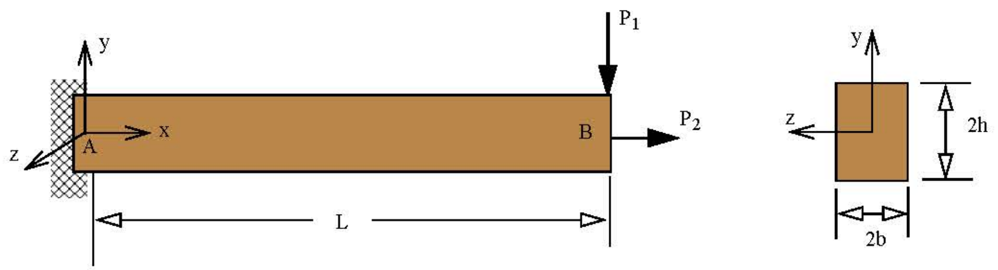
$$\sigma_{xx} = c_4x^2 + d_4xy - (a_4 + 2c_4)y^2$$

$$\sigma_{yy} = a_4x^2 + b_4xy + c_4y^2$$

$$\tau_{xy} = -\left(\frac{b_4}{2}x^2 + 2c_4xy + \frac{d_4}{2}y^2\right)$$

C6.8 Figure below shows a cantilever beam with a rectangular cross-section. An Airy stress function that could be used is given by equation below. In terms of P_1, P_2, b, h, x, y, and L, determine (a) the stress components σ_{xx}, σ_{yy}, and τ_{xy}. (b) the displacement components u_x, u_y. Assume plane stress.

$$\psi = a_1\left[3\frac{y}{h} - \left(\frac{y}{h}\right)^3\right](x-L) + a_2\left(\frac{y}{h}\right)^2$$



Torsion of non-circular shafts (prismatic bars)

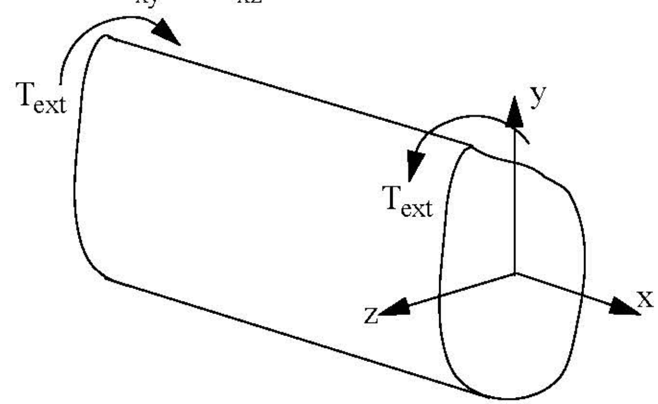
- We develop a more general theory in which there is no limitation on the cross-section shape or the thickness of the shaft.
- Saint-Venant was the first to develop the theory of torsion for non-circular shafts.
- Prandtl later developed an alternative based on Airy's stress functions.
- We will use Prandtl's approach to obtain the stresses and Saint Venant's approach to obtain the deformation.

Saint Venant's method (semi inverse method)

Saint Venant observed that a displacement field should account for the following:

(i) the cross-section would warp under torsion defined by a warping function: $\chi(y, z)$.

(ii) the only non-zero stress components would be τ_{xy} and τ_{xz}



1. **Displacements:** $u = \chi(y, z)\dfrac{d\phi}{dx}$ $\quad$ $v = -xz\dfrac{d\phi}{dx}$ $\quad$ $w = xy\dfrac{d\phi}{dx}$;

$\dfrac{d\phi}{dx}$ --- the rate of twist per unit length and is a constant for a segment of a shaft.

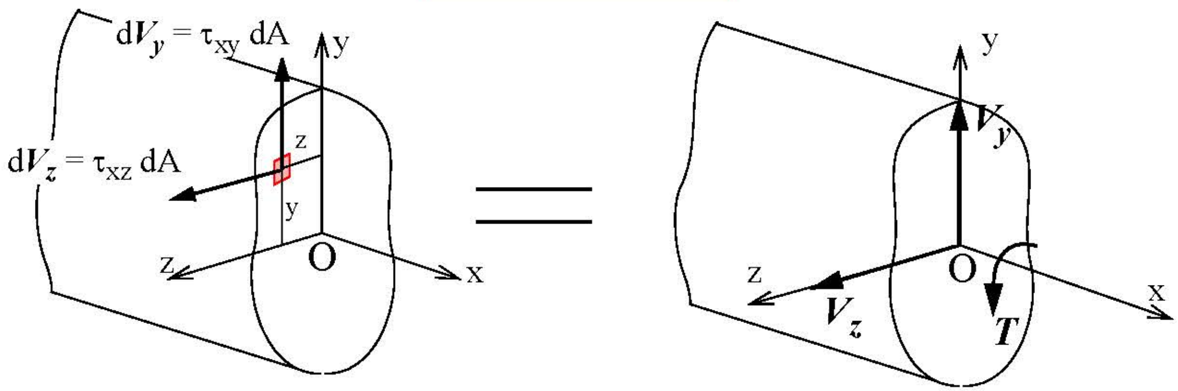
2. **Strains:** $$\gamma_{xy} = \frac{\partial}{\partial y}\left[\chi(y, z)\frac{d\phi}{dx}\right] + \frac{\partial}{\partial x}\left[-xz\frac{d\phi}{dx}\right] = \frac{d\phi}{dx}\left[\frac{\partial \chi}{\partial y} - z\right]$$

$$\gamma_{xz} = \frac{\partial}{\partial x}\left[xy\frac{d\phi}{dx}\right] + \frac{\partial}{\partial z}\left[\chi(y,z)\frac{d\phi}{dx}\right] = \frac{d\phi}{dx}\left[y + \frac{\partial \chi}{\partial z}\right]$$

3. **Stresses:** $\tau_{xy} = G\left(\frac{\partial \chi}{\partial y} - z\right)\frac{d\phi}{dx}$ $\qquad \tau_{xz} = G\left(\frac{\partial \chi}{\partial z} + y\right)\frac{d\phi}{dx}$

4. **Internal Torque**

$$\boxed{\boldsymbol{T} = \int_A (y\tau_{xz} - z\tau_{xy})\,dA}$$



- We could develop formulas as we did for other structural members, but we will use an alternative that requires Prandtl's method.

Prandtl's method

- All normal stresses are zero and shear stress $\tau_{yz} = 0$.

Equilibrium Equations: $\frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$ $\qquad \frac{\partial \tau_{xy}}{\partial x} = 0$ $\qquad \frac{\partial \tau_{xz}}{\partial x} = 0$

- τ_{xy}, τ_{xz} cannot be function of x. This requires no distributed torque on the shaft.

Prandtl defines a stress function.

$$\boxed{\tau_{yx} = \tau_{xy} = \frac{\partial \psi}{\partial z} \qquad \tau_{zx} = \tau_{xz} = -\left(\frac{\partial \psi}{\partial y}\right)}$$

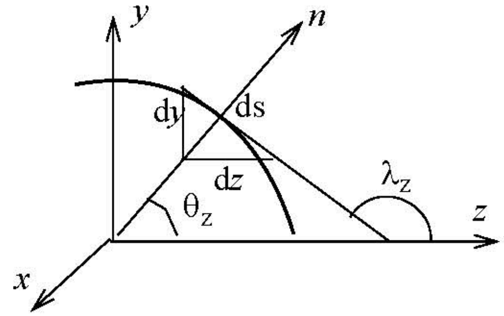
Boundary conditions:

- $S_y = 0, S_z = 0$ on the surface of the shaft are met by the stress state.

$$\tau_{xy}n_y + \tau_{xz}n_z = S_x = 0 \qquad \text{or} \qquad \frac{\partial\psi}{\partial z}n_y - \frac{\partial\psi}{\partial y}n_z = 0$$

$$n_y = -\left(\frac{dz}{ds}\right) \qquad n_z = \frac{dy}{ds}$$

$$-\left(\frac{\partial\psi}{\partial z}\right)\left(\frac{dz}{ds}\right) - \frac{\partial\psi}{\partial y}\left(\frac{dy}{ds}\right) = 0 \qquad \text{or} \qquad -\left(\frac{d\psi}{ds}\right) = 0$$



The stress function ψ must be constant on the boundary of the cross-section of the non-circular shaft

$$\boxed{T = 2\iint_A \psi\, dy\, dz}$$

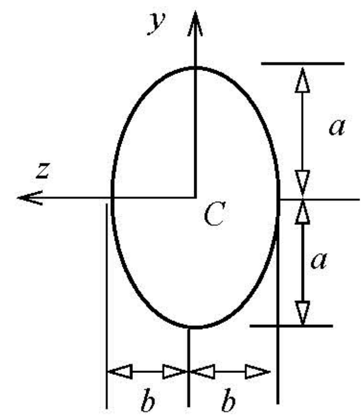
Option 1: From stresses, obtain strains, and integrate to get the displacements
Option 2: We equate Prandtl's stresses to St. Venant stresses.

$$\boxed{\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = -2G\frac{d\phi}{dx}}$$

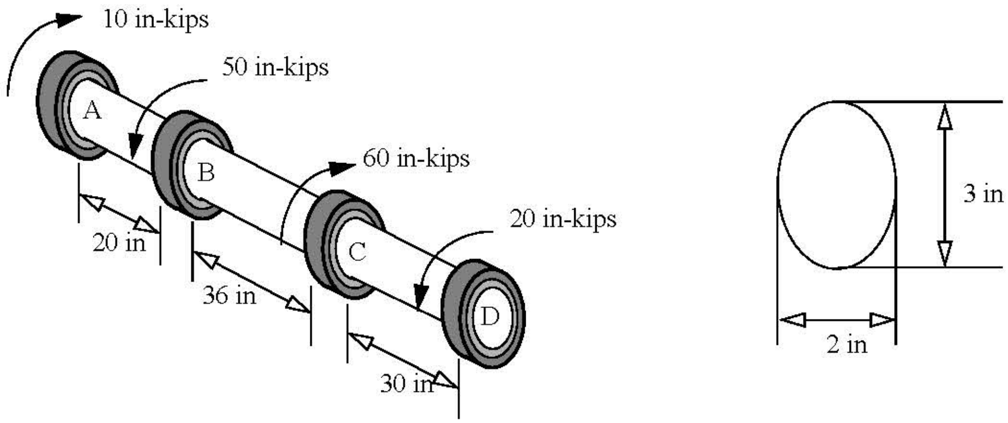
Procedure for solving problems of torsion of non-circular shafts

1. For a given cross-section shape obtain a stress function ψ that is constant on the boundary of the cross-section.
2. Determine any constant in the stress function in terms of the internal torque T by integrating over the cross-section.
3. Determine the Prandtl's shear stresses.
4. Determine the rate of twist $\frac{d\phi}{dx} = \frac{\Delta\phi}{\Delta x} = \frac{\phi_2 - \phi_1}{x_2 - x_1}$ and obtain $\Delta\phi = \phi_2 - \phi_1$.

C6.9 A shaft has an elliptical cross-section shown below. Determine the equations for maximum shear stress τ_{max} at a cross-section and relative rotation $(\phi_2 - \phi_1)$ of cross-sections at two points (x_1 and x_2) along the length of the shaft in terms of internal torque $\boldsymbol{T}$, shear modulus G, a, b, x_1 and x_2.



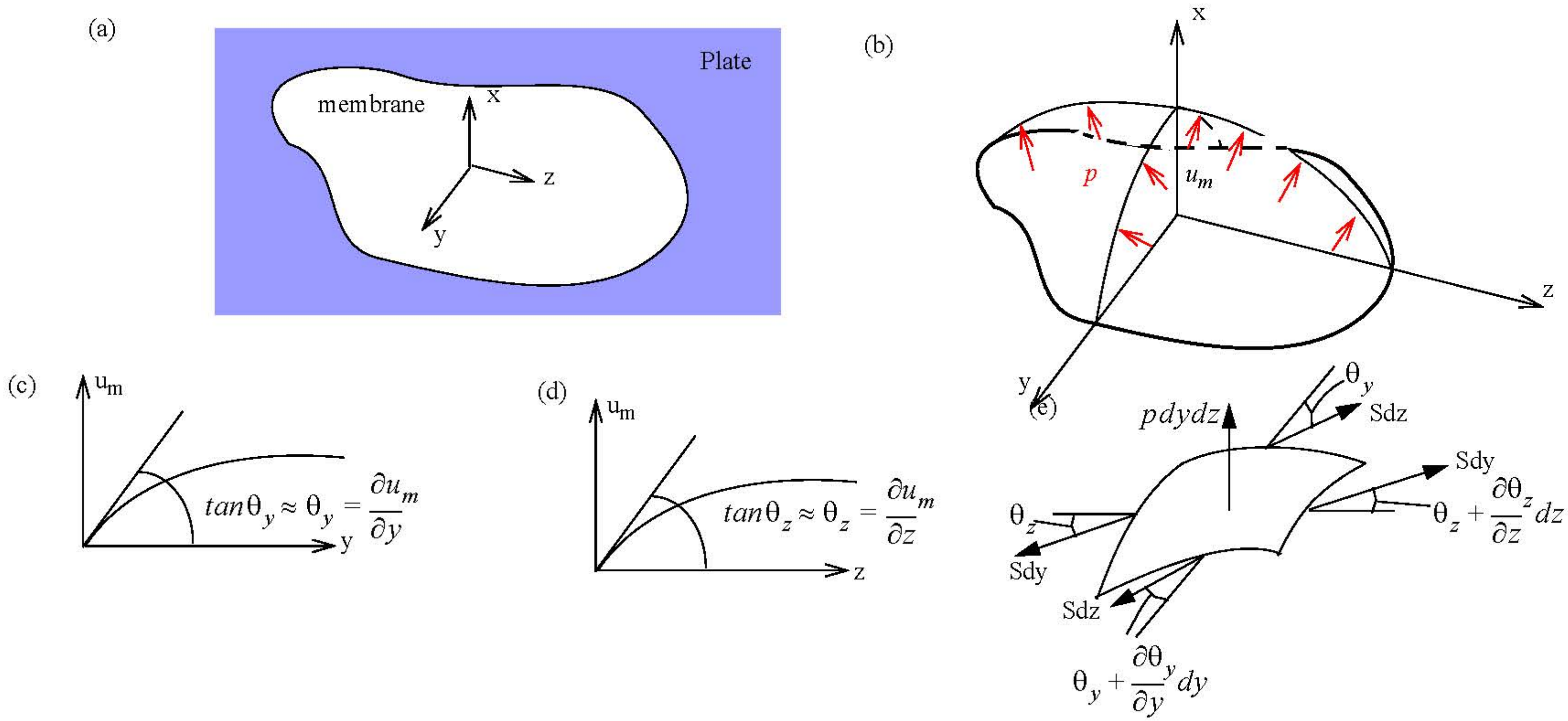
C6.10 An aluminum (G = 4000 ksi) elliptical shaft of is loaded as shown. Determine the maximum shear stress in the shaft and rotation of section D with respect to rotation of section at A.



Torsion membrane analogy for torsion of non-circular shafts

- Prandtl realized that the deflection of membrane under pressure has the same differential equation as torsion of non-circular bars.
- important in experimentally determining the torsional rigidity of complex cross sectional shapes such as cross section of a wing of an aircraft.

u_m ---membrane deflection. S ----uniform tension per unit length acts in the tangent direction of deflected membrane.



Differential Equation: $\frac{\partial^2 u_m}{\partial y^2} + \frac{\partial^2 u_m}{\partial z^2} = -\frac{p}{S}$

Boundary Condition: $u_m = 0$ on boundary

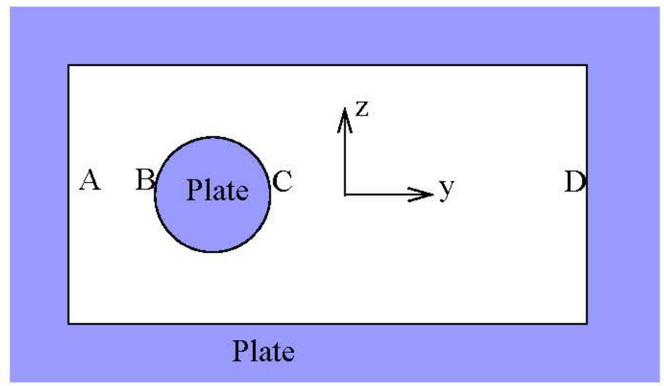
Membrane Analogy

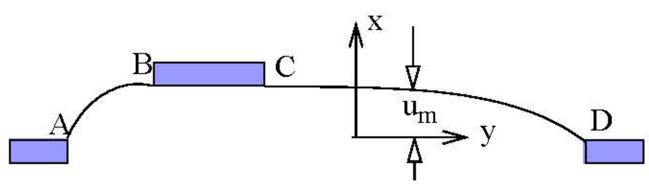
Membrane deflection problem	Prantdl's Torsion problem formulation
$\frac{\partial^2 u_m}{\partial y^2} + \frac{\partial^2 u_m}{\partial z^2} = -\frac{p}{S}$	$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -2G\frac{d\phi}{dx}$
u_m	ψ
p	$2\frac{d\phi}{dx}$
$\frac{1}{S}$	G
$\frac{\partial u_m}{\partial z}$	$\tau_{yx} = \tau_{xy} = \frac{\partial \psi}{\partial z}$
$\frac{\partial u_m}{\partial y}$	$\tau_{zx} = \tau_{xz} = -\left(\frac{\partial \psi}{\partial y}\right)$
Volume beneath the membrane $\iint_A u_m dy dz$	$\boldsymbol{T} = 2\iint_A \psi dy dz$

- The internal torque capacity of a cross-section can be found by doubling the volume under the deflected membrane, in other words torsional rigidity.
- Analogy can also be used to find the maximum torsional shear stress.

Membrane analogy for cross section with holes

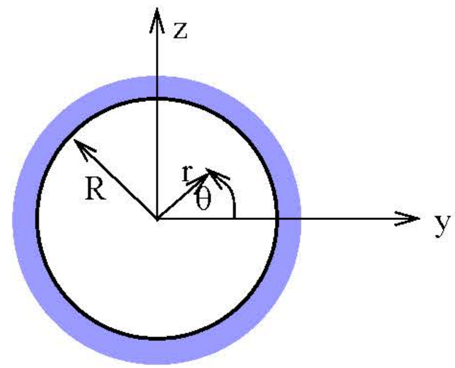
- No deformation in the hole and zero shear stress on hole boundary. Model the hole with a rigid plate such that slope at the hole boundary is zero.





C6.11

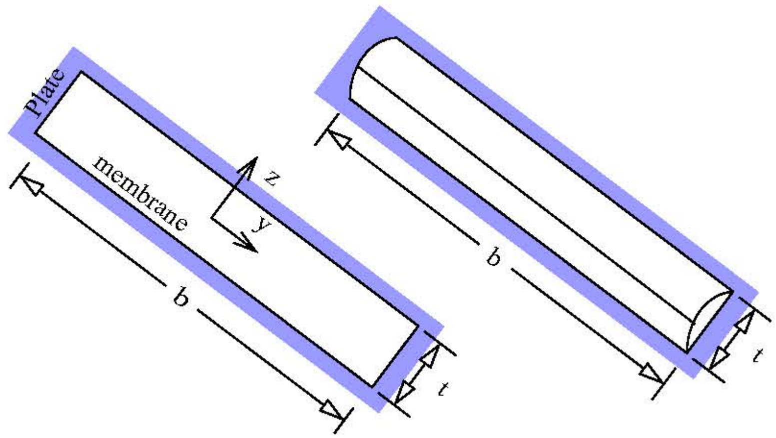
Obtain the formulas of torsion of circular shafts shown using membrane analogy.



Torsion of thin-walled open section

- Obtaining analytical formulas for torsion of thin-walled open sections is difficult.
- To develop the approximate formulas we will first consider a *thin* rectangular cross section. $b > 10t$
- The membrane deflection will be dominated by z and does not change significantly with *y*, i.e., $u_m(z)$.

$$\frac{d^2 u_m}{dz^2} = -\frac{p}{S}$$



$$u_m = -\frac{pz^2}{2S} + C_1 z + C_2 \qquad u_m(z = \pm t/2) = 0$$

$$u_m = \frac{p}{2S}\left(\frac{t^2}{4} - z^2\right)$$

- At $y = \pm b/2$ the approximate membrane deflected shape is incorrect.

Volume: $$\boxed{V = b \int_{-t/2}^{t/2} u_m \, dz = \frac{bpt^3}{12S}}$$

From Analogy: $V \Rightarrow \boldsymbol{T}/2 \qquad p \Rightarrow 2\frac{d\phi}{dx} \qquad \frac{1}{S} \Rightarrow G$

$$\boxed{\boldsymbol{T} = G\frac{bt^3}{3}\frac{d\phi}{dx} = GK\frac{d\phi}{dx} \qquad \text{where } K = bt^3/3}$$

- GK is the *torsional rigidity*. K is not the polar moment of inertia.
- The shear stress is related to the slope of the deflected membrane curve.

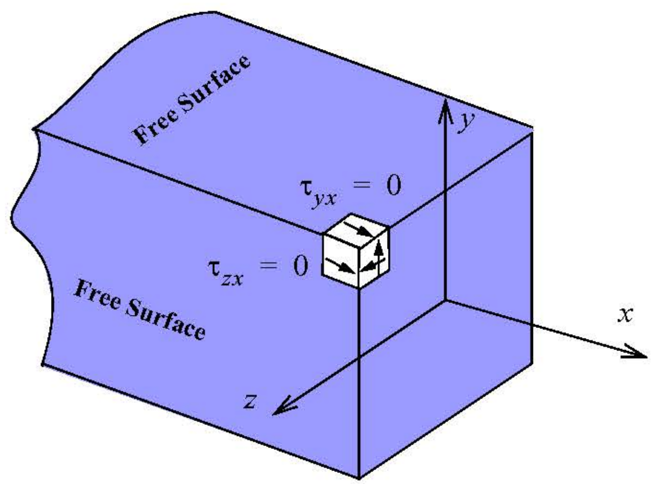
$$\frac{\partial u_m}{\partial z} = -\left(\frac{pz}{S}\right)$$

From analogy $\frac{\partial u_m}{\partial z} \Rightarrow \tau_{xy}$ $\quad p \Rightarrow 2\frac{d\phi}{dx}$ $\quad \frac{1}{S} \Rightarrow G$ $\quad \tau_{xy} = -2G\frac{d\phi}{dx}z$

- In thin-walled closed sections we assumed that the torsional shear stress in the thickness direction was uniform.
- In thin-walled open section, the torsional shear stress varies linearly in the thickness direction. The maximum shear stress will be at edge of the thickness, i.e., at $z = \pm t/2$.
- y is in the direction of the center line.

Maximum torsional shear stress: $\boxed{\tau = \frac{Tt}{K}}$

End effects in rectangular cross sections



- There is a transition region near towards the edge of $y = \pm b/2$ where the torsional shear stress τ_{xz}, which we neglected is non-zero.

C6.12 (a) Obtain the torsional rigidity and maximum shear stress for the thin open cross section of uniform thickness t shown below. Assume $t \ll a$ and gap at D is of negligible thickness. Report the answer in terms of internal torque $\boldsymbol{T}$, shear modulus G, thickness t, and parameter a. (b) Determine the torsional rigidity and maximum shear stress of thin closed section where the gap at D is closed. [See Eqs (6.32) and (6.34) of the book].