Stability

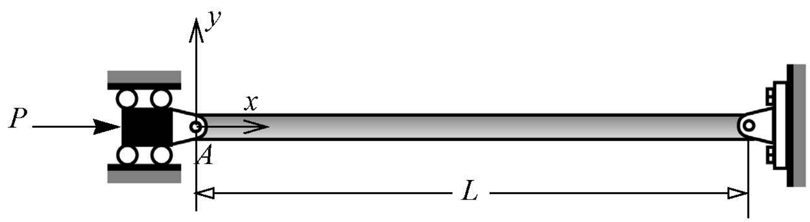
Learning objective

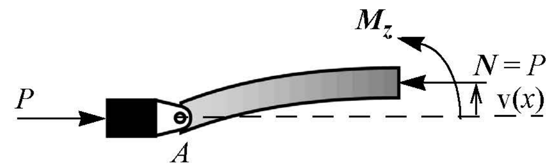
- Understand the concept of buckling of columns.

Definitions

- Bending due to a compressive axial load is called buckling.
- Structural members that support compressive axial loads are called columns.
- Buckling is the study of stability of a structure's equilibrium.
- Instability due to compressive inplane stresses of thin plates and shells is called local buckling.
- Instability of a structural member due to compressive inplane loads is called structural buckling.

Euler Buckling





Boundary Value Problem

Differential Equation: $EI\dfrac{d^2v}{dx^2} + Pv = 0$

Boundary conditions: $v(0) = 0 \qquad v(L) = 0$

Trivial Solution: $v = 0$

Non-Trivial Solution: $v(x) = A\cos\lambda x + B\sin\lambda x$ where $\lambda = \sqrt{\dfrac{P}{EI}}$

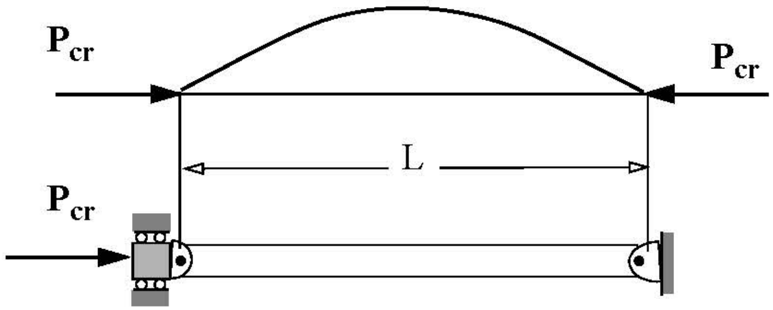
Characteristic Equation: $\sin\lambda L = 0$; $P_n = \dfrac{n^2\pi^2 EI}{L^2}$ $\qquad n = 1, 2, 3 \cdots$

Euler Buckling Load: $P_{cr} = \dfrac{\pi^2 EI}{L^2}$

- Buckling occurs about an axis that has a minimum value of I.

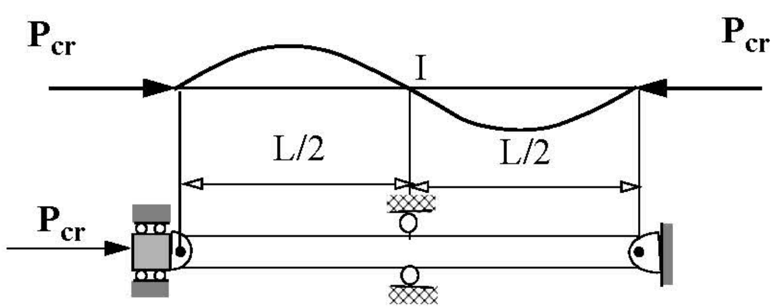
Buckling Mode: $v = B\sin\left(n\pi\frac{x}{L}\right)$

Mode shape 1



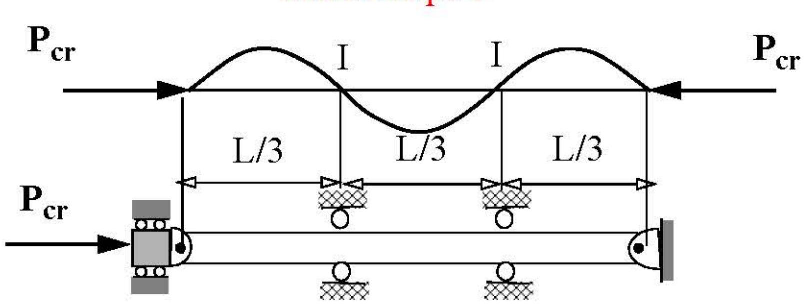
$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

Mode shape 2



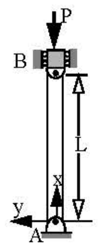
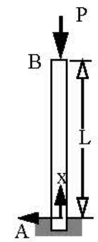
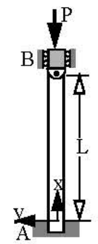
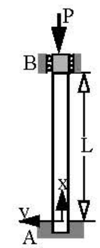
$$P_{cr} = \frac{4\pi^2 EI}{L^2}$$

Mode shape 3



$$P_{cr} = \frac{9\pi^2 EI}{L^2}$$

Effects of End Conditions

Case	1.	2.	3.	4.
Case	 Pinned at both Ends	 One end fixed, other end free	 One end fixed, other end pinned	 Fixed at both ends.
Differential Equation	$EI\frac{d^2v}{dx^2} + Pv = 0$	$EI\frac{d^2v}{dx^2} + Pv = Pv(L)$	$EI\frac{d^2v}{dx^2} + Pv = R_B(L-x)$	$EI\frac{d^2v}{dx^2} + Pv = R_B(L-x) + M_B$
Boundary Conditions	$v(0) = 0$ $v(L) = 0$	$v(0) = 0$ $\quad \frac{dv}{dx}(0) = 0$	$v(0) = 0$ $\quad \frac{dv}{dx}(0) = 0$ $v(L) = 0$	$v(0) = 0$ $\quad \frac{dv}{dx}(0) = 0$ $v(L) = 0$ $\quad \frac{dv}{dx}(L) = 0$
Characteristic Equation $\lambda = \sqrt{P/(EI)}$	$sin\lambda L = 0$	$cos\lambda L = 0$	$tan\lambda L = \lambda L$	$2(1 - cos\lambda L) - \lambda L sin\lambda L = 0$
Critical Load P_{cr}	$\frac{\pi^2 EI}{L^2}$	$\frac{\pi^2 EI}{4L^2} = \frac{\pi^2 EI}{(2L)^2}$	$\frac{20.13EI}{L^2} = \frac{\pi^2 EI}{(0.7L)^2}$	$\frac{4\pi^2 EI}{L^2} = \frac{\pi^2 EI}{(0.5L)^2}$ [
Effective Length—L_{eff}	L	$2L$	$0.7L$	$0.5L$

$$P_{cr} = \frac{\pi^2 EI}{L_{eff}^2}$$

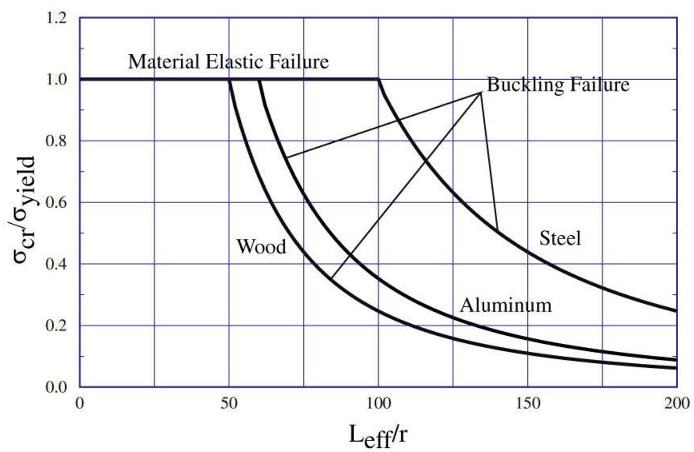
Classification of columns

Radius of gyration: $r = \sqrt{I/A}$

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(L_{eff}/r)^2}$$

Slenderness ratio: (L_{eff}/r).

Failure Envelopes

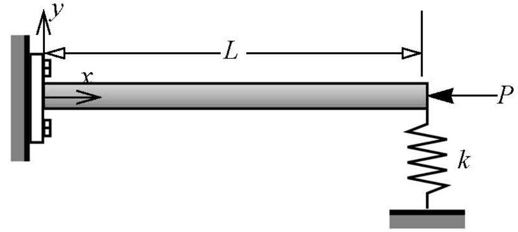


- Short columns: Designed to prevent material elastic failure.
- Long columns: Designed to prevent buckling failure.
- Intermediate column is a third classification used if the critical stress is between yield stress and ultimate stress.

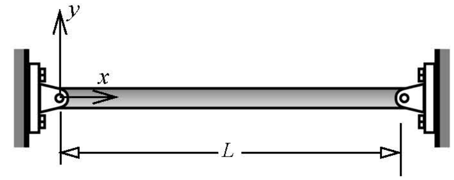
$$P_{\mathrm{cr}} = \frac{\pi^2 E_t I}{L_{\mathrm{eff}}^2}$$

where E_t is the tangent modulus, which depends on the stress level P_{cr}/A.

C4.1 Linear spring is attached at the free end of a column as shown below. Assume that bending about the y axis is prevented. (a) Determine the characteristic equation for this buckling problem. Show that the critical load P_{cr} for (b) $k = 0$ and (c) $k = \infty$ is as given in Table 4.1 for cases 2 and 3, respectively.



C4.2 In terms of α the thermal coefficient of expansion, r is the radius of gyration, and length L, determine the critical change of temperature at which the beam shown in Example 4.8 will buckle. Assume a uniform increase of temperature.

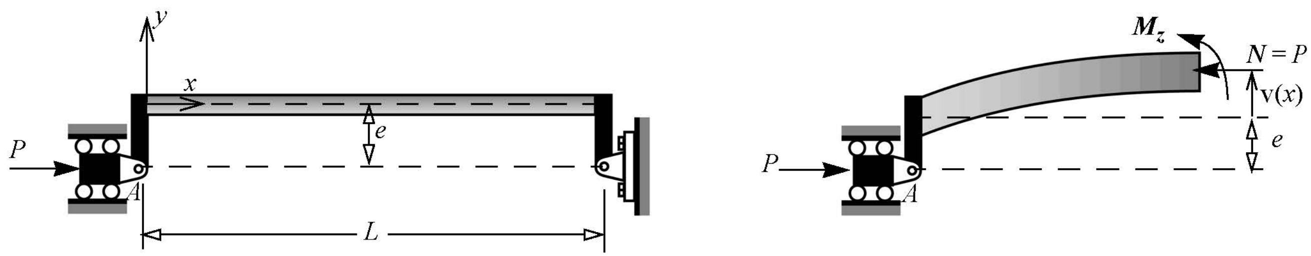


Imperfect columns

Imperfections

- The column material may contain small holes, minute cracks, or other material inclusions. Hence the homogeneity requirement or the requirement that the centroids of all cross sections be on a straight line may not be met.
- The material processing may cause local strain hardening. Hence the condition of linear and elastic material behavior across the entire cross section may not be met.
- The theoretical design centroid and the actual centroid are offset due to manufacturing tolerances.
- Local conditions at the support cause the reaction force to be offset from the centroid.
- The transfer of loads from one member to another may not occur at the centroid.

Imperfections in the column cause the application of axial loads to be offset from the centroid of the cross section. This offset loading is termed **eccentric loading** on columns.



Differential equation: $\dfrac{d^2\mathrm{v}}{dx^2} + \lambda^2\mathrm{v} = -\left(\dfrac{Pe}{EI}\right)$

Boundary Conditions: $\mathrm{v}(0) = 0 \qquad \mathrm{v}(L) = 0$

$$\mathrm{v}(x) = \mathrm{v}_H + \mathrm{v}_P = A\cos\lambda x + B\sin\lambda x - e$$

$$\mathrm{v}(x) = e[\cos\lambda x + \tan(\lambda L/2)\sin\lambda x - 1]$$

$$\boxed{v_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right]}$$

Secant Formula: $$\boxed{\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L_{eff}}{2r}\sqrt{\frac{P}{EA}}\right)\right]}$$

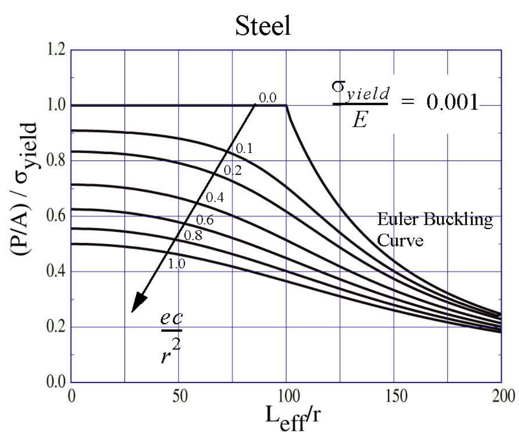
- The quantity ec/r^2 is called the *eccentricity ratio*.

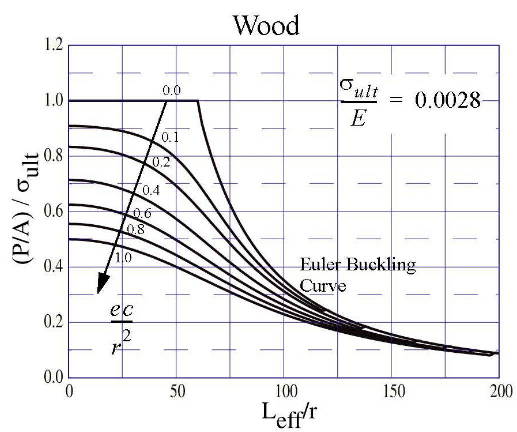
$$\frac{P/A}{\sigma_{fail}}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L_{eff}}{2r}\sqrt{\left(\frac{\sigma_{fail}}{E}\right)\frac{P/A}{\sigma_{fail}}}\right)\right] = 1$$

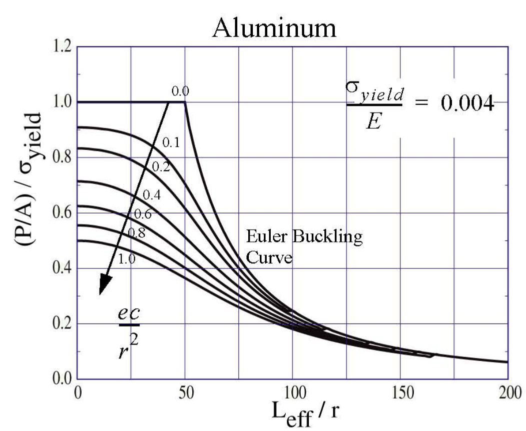
Failure Envelopes:

Steel and Aluminum: Failure stress is yield stress. $\sigma_{fail} = \sigma_{yield}$;

Wood: Failure stress is ultimate stress. $\sigma_{fail} = \sigma_{ult}$







C4.3 A wooden box column (E = 1800 ksi) is constructed by joining four pieces of lumber together, as shown below. The load P = 80 kips is applied at a distance of e = 0.667 in. from the centroid of the cross section. (a) If the length is L = 10 ft, what are the maximum stress and the maximum deflection? (b) If the allowable stress is 3 ksi, what is the maximum permissible length L to the nearest inch?