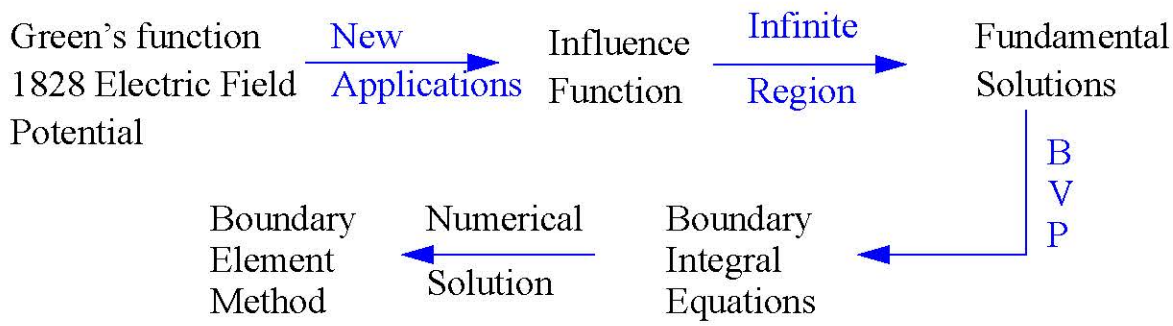
Influence Function for Beams

Learning Objective

- Understand the concept of influence functions and its applications to classical beams and beams on elastic foundations.

History



Mathematical Preliminaries

A **source point** (x) is a point in the material at which a disturbance is placed.

A **field point** (ξ) is a point in the material where the impact of disturbance is evaluated.

Influence function $G(x, \xi)$ relates a value of a variable at the field point to a unit value of the disturbance at the source point. In other words it evaluates the influence of a disturbance at field point.

Influence functions associated with infinite bodies are called **fundamental solutions**.

The influence function is said to be **singular** if it becomes infinite at the source point.

The disturbance associated with a singular influence function is called a **singularity**

Singularity represented by the delta function it is called the **source singularity**.

When two source singularities of equal and opposite magnitude are placed at infinitesimal distance that shrinks to zero such that magnitude of the resulting singularity is finite, then the new singularity is called the **doublet singularity**.

Force (Source) singularity influence functions in beams

In the differential equation:

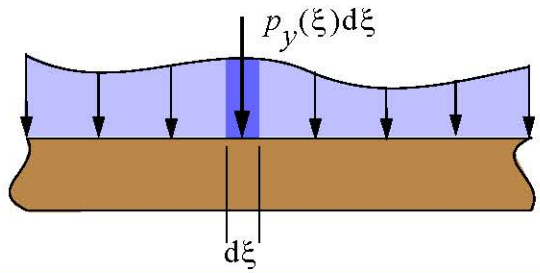
replace forcing function (p_y) by delta function

replace v(x) by $G(x, \xi)$

Solve BVP for $G(x, \xi)$

- Influence function $G(x, \xi)$ represents the deflection at point x on the beam due to a unit value of force placed at ξ.

$$\boxed{\mathrm{v}(x) = G(x, \xi)P} \qquad \boxed{\mathrm{v}(x) = \int_{Length} G(x, \xi)p_y(\xi)d\xi}$$



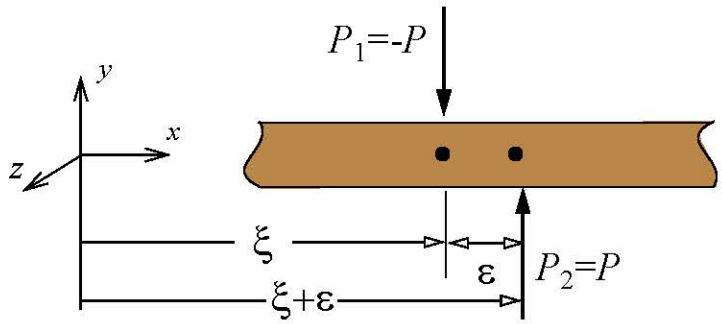
$$\boxed{\psi(x) = \frac{\mathrm{dv}}{\mathrm{d}x} = \int_{Length} G_1(x, \xi)p_y(\xi)d\xi \quad \text{where} \quad G_1(x, \xi) = \frac{\partial G}{\partial x}}$$

$$\boxed{M_z(x) = EI\frac{\mathrm{d}^2\mathrm{v}}{\mathrm{d}x^2} = \int_{Length} G_2(x, \xi)p_y(\xi)d\xi \quad \text{where} \quad G_2(x, \xi) = EI\frac{\partial G_1}{\partial x}}$$

$$\boxed{V_y(x) = -\left(\frac{\mathrm{d}M_z}{\mathrm{d}x}\right) = \int_{Length} G(x, \xi)p_y(\xi)d\xi \quad \text{where} \quad G_3(x, \xi) = -\left(\frac{\partial G_2}{\partial x}\right)}$$

- $G_1(x, \xi)$ is the slope at field point x due to a unit value of force placed at source point ξ.
- $G_2(x, \xi)$ is the internal bending moment at field point x due to a unit value of force placed at source point ξ.
- $G_3(x, \xi)$ is the internal shear force at field point x due to a unit value of force placed at source point ξ.

Moment (Doublet) singularity influence functions in beams



$$\mathrm{v}(x) = G(x,\xi)P_1 + G(x,\xi+\varepsilon)P_2 = -G(x,\xi)P + G(x,\xi+\varepsilon)P$$

$$\mathrm{v}(x) = -G(x,\xi)P + \left[G(x,\xi) + \frac{\partial G}{\partial \xi}\varepsilon + \frac{1}{2!}\frac{\partial^2 G}{\partial \xi^2}\varepsilon^2 + \bullet\ \bullet\right]P = P\varepsilon\left[\frac{\partial G}{\partial \xi} + \frac{1}{2!}\frac{\partial^2 G}{\partial \xi^2}\varepsilon + \bullet\ \bullet\right]$$

Let $\lim\limits_{\varepsilon \to 0}\lim\limits_{P \to \infty}(P\varepsilon) = M$ to obtain

$$\boxed{\mathrm{v}(x) = M\left[\frac{\partial G}{\partial \xi}\right] = H(x,\xi)M \qquad H(x,\xi) = \frac{\partial G}{\partial \xi}}$$

- M is *positive counterclockwise* with respect to the z-axis.

$$\boxed{\psi(x) = \frac{\mathrm{dv}}{\mathrm{d}x} = \int_{Length} H_1(x,\xi)p_y(\xi)d\xi \qquad \text{where} \qquad H_1(x,\xi) = \frac{\partial H}{\partial x}}$$

$$\boxed{M_z(x) = EI\frac{\mathrm{d}^2\mathrm{v}}{\mathrm{d}x^2} = \int_{Length} H_2(x,\xi)p_y(\xi)d\xi \qquad \text{where} \qquad H_2(x,\xi) = EI\frac{\partial H_1}{\partial x}}$$

$$\boxed{V_y(x) = -\left(\frac{\mathrm{d}M_z}{\mathrm{d}x}\right) = \int_{Length} H(x,\xi)p_y(\xi)d\xi \qquad \text{where} \qquad H_3(x,\xi) = -\left(\frac{\partial H_2}{\partial x}\right)}$$

- $H(x,\xi)$ is the displacement at field point x due to a unit value of moment placed at source point ξ.
- $H_1(x,\xi)$ is the slope at field point x due to a unit value of moment placed at source point ξ.
- $H_2(x,\xi)$ is the internal bending moment at field point x due to a unit value of moment placed at source point ξ.
- $H_3(x,\xi)$ is the internal shear force at field point x due to a unit value of moment placed at source point ξ.

Numerical Integration

$$I = \int_a^b f(x)dx = \sum_{i=1}^{N+1} w_i f(x_i)$$

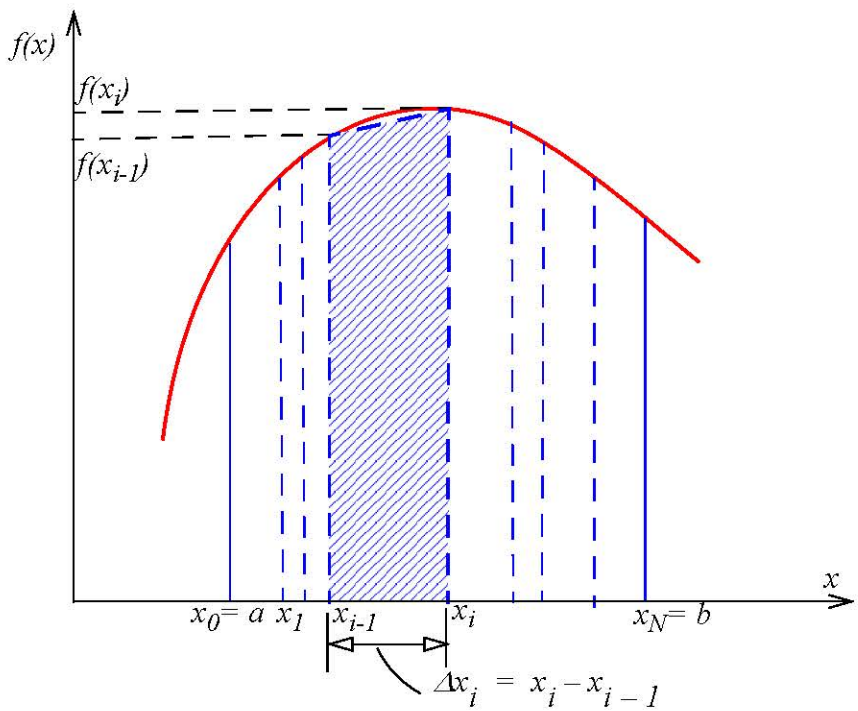
x_i are called the base points. For N+1 base points divides the (b-a) into N intervals.

w_i are called the weights.

- The choice of base points and weights define various integration schemes. Gauss quadrature is the preferred integration scheme in most computer programs as it gives excellent accuracies for *smooth integrands*.

Trapezoidal Rule

- Simplest numerical integration scheme



$$I \cong \sum_{i=1}^{N} \frac{\Delta x_i}{2}[f(x_i) + f(x_{i-1})]$$

Equally spaced base points

$$I \cong \left[\sum_{i=1}^{N-1} f(x_i) + \frac{f(x_0) + f(x_N)}{2}\right]\Delta x$$

Non-dimensional variables

- Improves numerical accuracy
- Makes algorithms independent of units.
- Simplifies computer programing.

Notation: Variables with curved bars will refer to the non-dimensional variable. $\widehat{x} = x/L_o$ $\widehat{\xi} = \xi/L_o$ $\widehat{p_y} = p_y/p_o$

- $p_o L_o$ has the dimension of force;
- $p_o L_o^{\,2}$ has the dimension of moment;
- $p_o L_o^{\,3}/EI$ has the dimension of slope;
- $p_o L_o^{\,4}/EI$ has the dimension of deflection.

$$[\widehat{\mathrm{v}}(\widehat{x})](p_o L_o^{\,4}/EI) = G(x,\xi)[\widehat{P}\,p_o L_o]$$

$$\widehat{\mathrm{v}}(\widehat{x}) = [G(x,\xi)(EI/L_o^{\,3})]\widehat{P} = \widehat{G}(\widehat{x},\widehat{\xi})\widehat{P}$$

Original variable	Non-dimensionalized variable
x	$\widehat{x} = x/L_o$
ξ	$\widehat{\xi} = \xi/L_o$
p_y	$\widehat{p_y} = p_y/p_o$
P	$\widehat{P} = P/(p_o L_o)$
v	$\widehat{\mathrm{v}} = \mathrm{v}[EI/(p_o L_o^{\,4})]$
$\psi = \dfrac{\mathrm{dv}}{\mathrm{d}x}$	$\widehat{\psi} = \psi[EI/(p_o L_o^{\,3})]$
M	$\widehat{M} = M/(p_o L_o^{\,2})$
$M_z = EI\dfrac{\mathrm{d}^2\mathrm{v}}{\mathrm{d}x^2}$	$\widehat{M_z} = M_z/(p_o L_o^{\,2})$
$V_y = -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(EI\dfrac{\mathrm{d}^2\mathrm{v}}{\mathrm{d}x^2}\right)$	$\widehat{V_y} = V_y/(p_o L_o)$

Original variable	Non-dimensionalized variable
G	$\widehat{G} = G[EI/L_o^{\,3}]$
$G_1 = \dfrac{\partial G}{\partial x}$	$\widehat{G}_1 = G_1[EI/L_o^{\,2}] = \dfrac{\partial \widehat{G}}{\partial \widehat{x}}$
G_2	$\widehat{G}_2 = G_2/L_o = \dfrac{\partial \widehat{G}_1}{\partial \widehat{x}}$
G_3	$\widehat{G}_3 = G_3 = \dfrac{\partial \widehat{G}_2}{\partial \widehat{x}}$
H	$\widehat{H} = H[EI/L_o^{\,2}]$
$H_1 = \dfrac{\partial H}{\partial x}$	$\widehat{H}_1 = H_1[EI/L_o] = \dfrac{\partial \widehat{H}}{\partial \widehat{x}}$
H_2	$\widehat{H}_2 = H_2 = \dfrac{\partial \widehat{H}_1}{\partial \widehat{x}}$
H_3	$\widehat{H}_3 = H_3 L_o = \dfrac{\partial \widehat{H}_2}{\partial \widehat{x}}$

Influence functions in classical beams

- Differential equation:

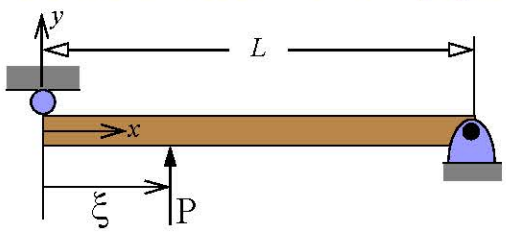
$$EI\frac{d^4G}{dx^4} = p_y = \langle x-\xi\rangle^{-1}$$

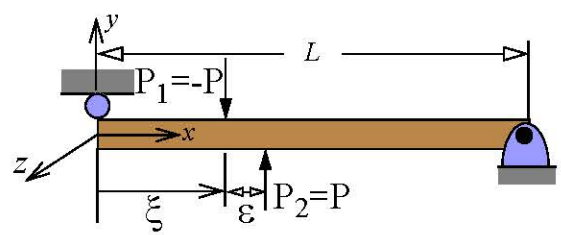
The homogeneous $G_h(x,\xi)$ and the particular solution $G_p(x,\xi)$ to the above equation are

$$G(x,\xi) = G_h(x,\xi) + G_p(x,\xi) \qquad G_h(x,\xi) = c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4 \qquad G_p(x,\xi) = \frac{1}{6EI}\langle x-\xi\rangle^3$$

$$\boxed{EIG(x,\xi) = \frac{1}{6}\langle x-\xi\rangle^3 + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4}$$

Influence function for simply supported beam





$$G(0,\xi) = 0 \qquad EI\frac{d^2G}{dx^2}(0,\xi) = 0$$

$$G(L,\xi) = 0 \qquad EI\frac{d^2G}{dx^2}(L,\xi) = 0$$

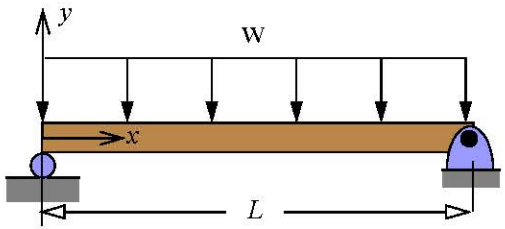
$$\boxed{G(x,\xi) = \frac{1}{6EI}\left[\langle x-\xi\rangle^3 - \frac{(L-\xi)}{L}x^3 - \frac{(L-\xi)^3}{L}x + (L-\xi)Lx\right] \qquad H(x,\xi) = \frac{\partial G}{\partial \xi} = \frac{1}{6EI}\left[-3\langle x-\xi\rangle^2 + \frac{x^3}{L} + \frac{3(L-\xi)^2}{L}x - Lx\right]}$$

- M is *positive counterclockwise* with respect to the z-axis.
- The Influence function incorporates the boundary conditions of a specific beam.

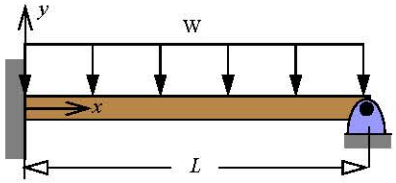
Non-dimensional influence functions

$$\widehat{G} = \frac{1}{6}[\langle \widehat{x} - \widehat{\xi}\rangle^3 - (1-\widehat{\xi})\widehat{x}^3 - (1-\widehat{\xi})^3\widehat{x} + (1-\widehat{\xi})\widehat{x}] \qquad \widehat{H} = \frac{1}{6}[-3\langle \widehat{x} - \widehat{\xi}\rangle^2 + \widehat{x}^3 + 3(1-\widehat{\xi})^2\widehat{x} - \widehat{x}]$$

C3.1 Obtain the elastic curve v(x) for the beam and loading shown



C3.2 Obtain the elastic curve v(x) for the beam shown.



Recollect

$$\boxed{G(x,\xi) = \frac{1}{6EI}\left[\langle x-\xi\rangle^3 - \frac{(L-\xi)}{L}x^3 - \frac{(L-\xi)^3}{L}x + (L-\xi)Lx\right]}$$

$$\widehat{G}(\widehat{x},\widehat{\xi}) = G(EI/L^3) = \frac{1}{6}[\langle \widehat{x}-\widehat{\xi}\rangle^3 - (1-\widehat{\xi})\widehat{x}^3 - (1-\widehat{\xi})^3\widehat{x} + (1-\widehat{\xi})\widehat{x}]$$

$$\widehat{\mathrm{v}}(\widehat{x}) = \mathrm{v}(\widehat{x})[EI/p_oL^4] = \sum_{i=1}^{N}\int_{\widehat{\xi}_{i-1}}^{\widehat{\xi}_i} \widehat{G}(\widehat{x},\widehat{\xi})\widehat{p_y}(\widehat{\xi})d\widehat{\xi}$$

Equally spaced base points: $\widehat{\mathrm{v}} = \left[\sum_{i=1}^{N-1}\widehat{G}_i\widehat{p}_i + \frac{1}{2}(\widehat{G}_0\widehat{p}_0 + \widehat{G}_N\widehat{p}_N)\right](\Delta\widehat{\xi})$

$$\widehat{M}_z = M_z/(p_oL^2) \qquad \widehat{V}_y = V_y/(p_oL)$$

$$\widehat{G} = \frac{1}{6}[\langle \widehat{x}-\widehat{\xi}\rangle^3 - (1-\widehat{\xi})\widehat{x}^3 - (1-\widehat{\xi})^3\widehat{x} + (1-\widehat{\xi})\widehat{x}]$$

$$\widehat{G}_1 = \frac{\partial\widehat{G}}{\partial\widehat{x}} = \frac{1}{6}[3\langle \widehat{x}-\widehat{\xi}\rangle^2 - 3(1-\widehat{\xi})\widehat{x}^2 - (1-\widehat{\xi})^3 + (1-\widehat{\xi})]$$

$$\widehat{G}_2 = \frac{\partial\widehat{G}_1}{\partial\widehat{x}} = \langle \widehat{x}-\widehat{\xi}\rangle^1 - (1-\widehat{\xi})\widehat{x}$$

$$\widehat{G}_3 = \frac{\partial\widehat{G}_2}{\partial\widehat{x}} = \langle \widehat{x}-\widehat{\xi}\rangle^0$$

C3.3 A 10 ft. simply supported beam as shown in below is loaded with a distributed force whose values vary as shown in Table 1.1. The beam has a rectangular cross section with depth of 8 in. in the y direction and width of 2 in. in the z direction. The modulus of elasticity for of the beam material is 8,100 ksi. Determine (a) the deflection and bending moment at the mid section of the beam. (b) the maximum deflection and bending normal stress in the beam.

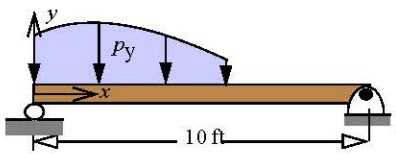


Table 1.1 Distributed load values.

x (ft.)	0	1	2	3	4	5	6	7	8	9	10
$p_y(x)$ (lb/ft)	-300	-310	-306	-288	-256	-210	0	0	0	0	0

$$I = \frac{1}{12}(2)(8^3) = 85.33 \text{ in.}^4 \qquad EI = (8100)(10^3)(85.33) = 691.2(10^6) \text{ lbs-in}^2 \qquad L_o = 10 \text{ ft.} \qquad p_o = p_{max} = 310 \text{ lb/ft}$$

$$\widehat{G} = [\langle \widehat{x} - \widehat{\xi} \rangle^3 - (1 - \widehat{\xi})\widehat{x}^3 - (1 - \widehat{\xi})^3 \widehat{x} + (1 - \widehat{\xi})\widehat{x}]/6 \qquad \widehat{G}_2 = [\langle \widehat{x} - \widehat{\xi} \rangle^1 - (1 - \widehat{\xi})\widehat{x}]$$

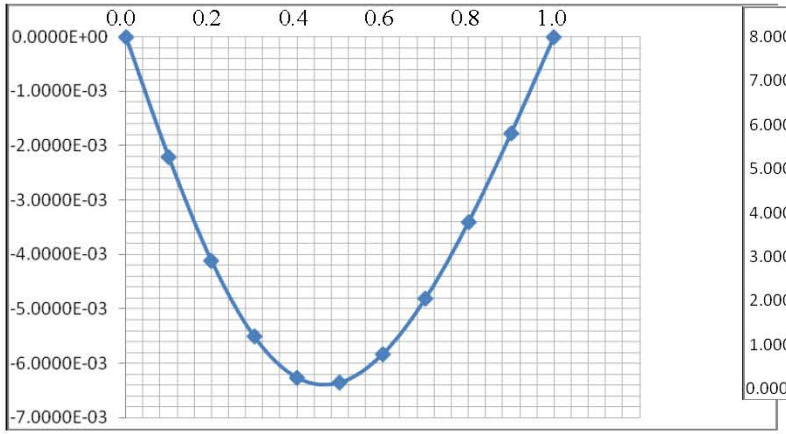
Solution at midpoint using spreadsheet

	A	B	C	D	E	F	G	H	I	J
1		x =	0.5							
2	ξ ft.	p_y lb/ft	$\widehat{\xi}$	$\widehat{p}_y$	$\widehat{x}$	$\langle \widehat{x} - \widehat{\xi} \rangle$	$\widehat{G}$	$\widehat{G}_i \widehat{p}_i$	$\widehat{G}_2$	$\widehat{G}_{2i} \widehat{p}_i$
3	0	-300	0	-0.9677	0.5	0.5	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
4	1	-310	0.1	-1.0000	0.5	0.4	6.1667E-03	-6.1667E-03	-5.0000E-02	5.0000E-02
5	2	-306	0.2	-0.9871	0.5	0.3	1.1833E-02	-1.1681E-02	-1.0000E-01	9.8710E-02
6	3	-288	0.3	-0.9290	0.5	0.2	1.6500E-02	-1.5329E-02	-1.5000E-01	1.3935E-01
7	4	-256	0.4	-0.8258	0.5	0.1	1.9667E-02	-1.6241E-02	-2.0000E-01	1.6516E-01
8	5	-210	0.5	-0.6774	0.5	0	2.0833E-02	-1.4113E-02	-2.5000E-01	1.6935E-01
9	6	0	0.6	0.0000	0.5	0	1.9667E-02	0.0000E+00	-2.0000E-01	0.0000E+00
10	7	0	0.7	0.0000	0.5	0	1.6500E-02	0.0000E+00	-1.5000E-01	0.0000E+00
11	8	0	0.8	0.0000	0.5	0	1.1833E-02	0.0000E+00	-1.0000E-01	0.0000E+00
12	9	0	0.9	0.0000	0.5	0	6.1667E-03	0.0000E+00	-5.0000E-02	0.0000E+00
13	10	0	1	0.0000	0.5	0	6.9389E-18	0.0000E+00	-5.5511E-17	0.0000E+00
14								-6.3530E-03		6.2258E-02

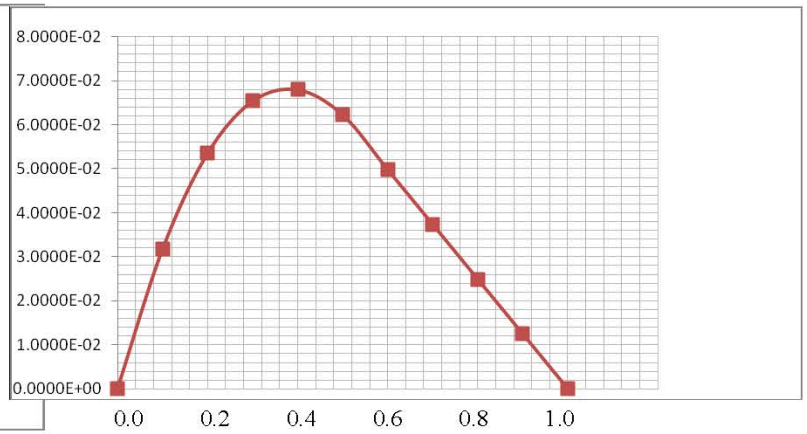
$$\mathrm{v}(x=5) = \frac{p_o L^4}{EI}\widehat{\mathrm{v}} = \frac{(25.833)(120^4)}{691.2(10^6)}[-6.3530(10^{-3})] = -49.235(10^{-3})\text{ in.}$$

$$\boldsymbol{M}_z(x=5) = p_o L^2 \widehat{\boldsymbol{M}}_z = (25.833)(120^2)(62.258)(10^{-3}) = 23160\text{ in.-lb}$$

Deflection



Bending moment



$$\widehat{\mathrm{v}}(x=0.5) = -6.3530(10^{-3}) \qquad \widehat{\boldsymbol{M}}_{max} = \widehat{\boldsymbol{M}}_z(x=0.4) = 67.93$$

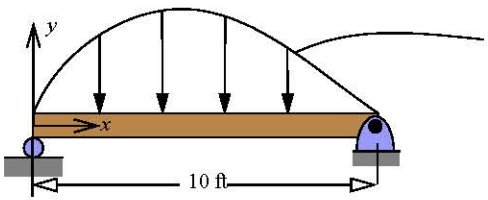
$$\mathrm{v}_{max} = \frac{p_o L^4}{EI}\widehat{\mathrm{v}}_{max} = \frac{(25.833)(120^4)}{691.2(10^6)}[-6.3530(10^{-3})] = -49.235(10^{-3})\text{ in.}$$

$$\boldsymbol{M}_{max} = p_o L^2 \widehat{\boldsymbol{M}}_{max} = (25.833)(120^2)(67.935)(10^{-3}) = 25272\text{ in.-lb}$$

$$(\sigma_{xx})_{max} = -\left[\frac{\boldsymbol{M}_{max} y_{max}}{I}\right] = -\left[\frac{(25272)(\pm 4)}{85.33}\right] = \mp 1184.7\text{ psi}$$

C3.4

A 10 ft. simply supported beam shown in Figure 1.1 has a distributed force that varies as shown. The beam has a rectangular cross section with depth of 8 in. in the y direction and width of 2 in. in the z-direction. The modulus of elasticity for of the beam material is 8,100 ksi. Determine (a) the deflection and bending moment at the mid section of the beam. (b) the maximum deflection and bending normal stress in the beam.



$$p_y = 300e^{-(x/10)} sin(\pi x/10) \text{ lb/ft.}$$

Solution: Convert the distributed function to numerical values and solve it as in previous example.

Table 1.2 Value of distributed load

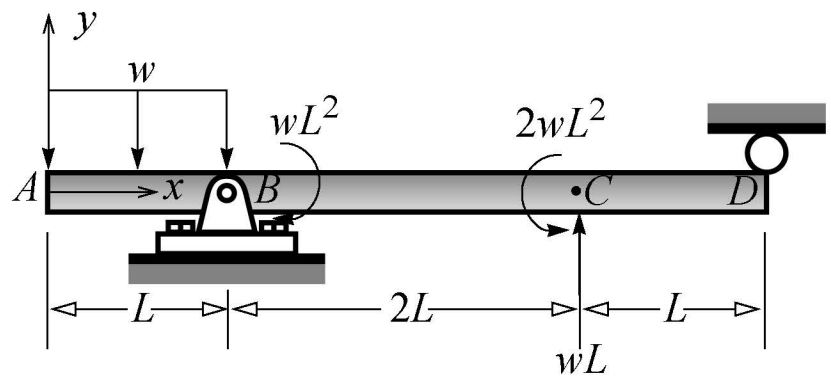
x (ft)	$p_y(x)$ (lb/ft)	x (ft)	$p(x)$ (lb/ft)
0	0	6	-156.585
1	-83.883	7	-120.523
2	-144.371	8	-79.233
3	-179.800	9	-37.691
4	-191.253	10	0.000
5	-181.959		

Class Problem 1

Using the influence functions associated with simply supported beams, write the equations for displacement and moment for the beam and loading shown. (Do not solve)

$$G(x,\xi) = \frac{1}{6EI}\left[\langle x-\xi\rangle^3 - \frac{(L-\xi)}{L}x^3 - \frac{(L-\xi)^3}{L}x + (L-\xi)Lx\right]$$

$$H(x,\xi) = \frac{1}{6EI}\left[-3\langle x-\xi\rangle^2 + \frac{x^3}{L} + \frac{3(L-\xi)^2}{L}x - Lx\right]$$

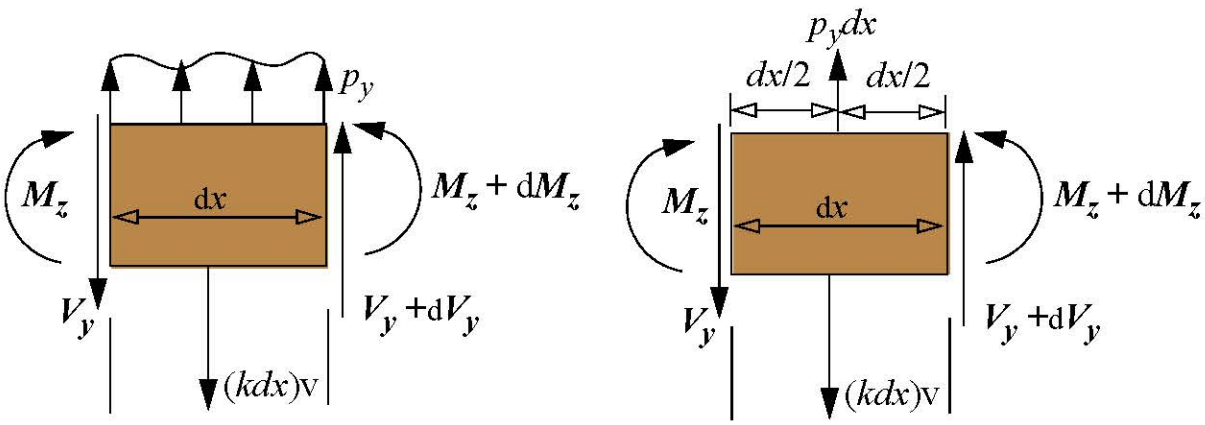


Beams on elastic foundations

- Railroad tracks rest on elastic support made of cross ties and earth,
- Long steel pipes rest on earth or a series of periodic supports,
- Load bearing walls in buildings have beams supported periodically by studs that rest on foundation beams.
- Machine frame made of beams resting on floors.

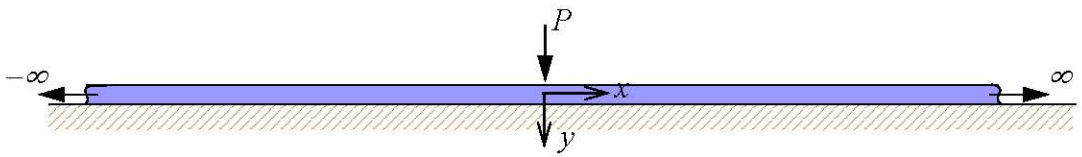
Winkler model

- The foundation resistance is assumed proportional to the beam deflection. This linear model works well for small deflection—the basic assumption in our beam theory.
- **modulus of foundation** k, is spring constant per unit length and has the dimension of force per length square.



$$\boxed{\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2 v}{dx^2}\right) + kv = p_y}$$

Fundamental solutions for beams on elastic foundations



Boundary Value Problem

Differential Equation: $$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2 v}{dx^2}\right) + kv = P\langle x\rangle^{-1}$$

Boundary Conditions: Disturbance due to the force dies out at infinity

$$\lim_{|x|\to\infty}\left[\frac{d}{dx}\left(EI_{zz}\frac{d^2 v}{dx^2}\right)\right] \to 0 \qquad \lim_{|x|\to\infty}\left[EI_{zz}\frac{d^2 v}{dx^2}\right] \to 0 \qquad \lim_{|x|\to\infty}[v] < \infty \qquad \lim_{|x|\to\infty}\left[\frac{dv}{dx}\right] < \infty$$

Integrating the differential equation from minus infinity to plus infinity we obtain

$$\int_{-\infty}^{\infty}\left[\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2 v}{dx^2}\right) + kv\right]dx = P\int_{-\infty}^{\infty}\langle x\rangle^{-1}dx$$

$$\frac{d}{dx}\left(EI_{zz}\frac{d^2 v}{dx^2}\right)\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} kv\,dx = P$$

Static Equilibrium: $$2\int_0^{\infty} kv\,dx = P$$

By symmetry: $\frac{dv}{dx}(0) = 0$ replace moment condition at infinity.

Consider solution for $x > 0$. In which case right hand side is zero.

Substitute $v = Ke^{\lambda x}$ into $\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2 v}{dx^2}\right) + kv = 0$ to obtain

$$K\left(\lambda^4 + \frac{k}{EI}\right)e^{\lambda x} = 0 \qquad \text{or} \qquad \lambda^4 + \frac{k}{EI} = 0$$

$$\boxed{\beta = \left(\frac{k}{4EI}\right)^{1/4}}$$

$$\lambda_1 = \beta(1+i) \qquad \lambda_2 = -\beta(1+i) \qquad \lambda_3 = \beta(1-i) \qquad \lambda_4 = -\beta(1-i) \qquad i = \sqrt{-1}$$

The 3rd and 4th roots are complex conjugate of roots 1 and 2.

$$v = \boldsymbol{Re}\{A_1 e^{\beta(1+i)x} + A_2 e^{-\beta(1+i)x}\} \qquad A_1 = A - iB \qquad A_2 = C + iD$$

$$\boxed{v = e^{\beta x}[A\cos\beta x + B\sin\beta x] + e^{-\beta x}[C\cos\beta x + D\sin\beta x] \qquad x > 0}$$

Condition: $x \to \infty$ the displacement v should remain bounded.

$$A = B = 0$$

$$v(x) = e^{-\beta x}[C\cos\beta x + D\sin\beta x] \qquad x > 0$$

Zero slope at origin $\quad C = D$

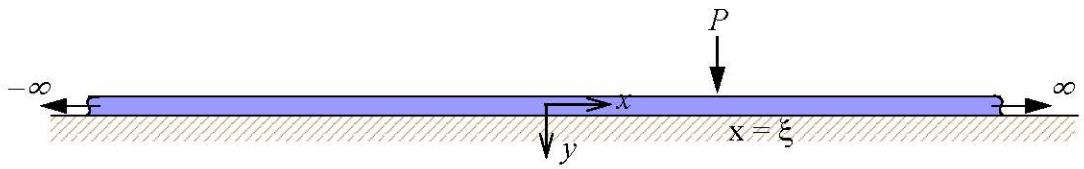
$$v(x) = Ce^{-\beta x}[\cos\beta x + \sin\beta x] \qquad x > 0$$

Equilibrium equation:

$$C = \left(\frac{P\beta}{2k}\right)$$

$$\boxed{v(x) = \left(\frac{P\beta}{2k}\right)e^{-\beta x}[\cos\beta x + \sin\beta x] \qquad x \geq 0}$$

Generalization:



Deflection:

$$\boxed{v(x) = G(x,\xi)P \qquad G(x,\xi) = \left(\frac{\beta}{2k}\right)e^{-\beta|x-\xi|}[\cos\beta|x-\xi| + \sin\beta|x-\xi|]}$$

***Sign function*:**

$$|x-\xi| = \begin{cases} (x-\xi) & x > \xi \\ -(x-\xi) & x < \xi \end{cases} \qquad \therefore \qquad \frac{\partial}{\partial x}|x-\xi| = \begin{cases} 1 & x > \xi \\ -1 & x < \xi \end{cases}$$

$$\boxed{sgn(x-\xi) = \begin{cases} 1 & x > \xi \\ -1 & x < \xi \end{cases}}$$

$$\frac{\partial}{\partial x}|x-\xi| = -\frac{\partial}{\partial \xi}|x-\xi| = sgn(x-\xi) = \begin{cases} 1 & x > \xi \\ -1 & x < \xi \end{cases}$$

$$\boxed{\frac{\partial G}{\partial x} = -\left(\frac{\partial G}{\partial \xi}\right) \qquad \text{Only valid for fundamental solutions.}}$$

Consider

$$H(x,\xi) = \frac{\partial G}{\partial \xi} \qquad G_1(x,\xi) = \frac{\partial G}{\partial x}$$

$$H(x,\xi) = -G_1(x,\xi)$$

Fundamental Solutions
$G = \left(\frac{\beta}{2k}\right)e^{-\beta\lvert x-\xi\rvert}[\cos\beta\lvert x-\xi\rvert + \sin\beta\lvert x-\xi\rvert]$
$G_1 = -\left(\frac{\beta^2}{k}\right)sgn(x-\xi)e^{-\beta\lvert x-\xi\rvert}\sin\beta\lvert x-\xi\rvert$
$G_2 = \left(\frac{1}{4\beta}\right)e^{-\beta\lvert x-\xi\rvert}(\sin\beta\lvert x-\xi\rvert - \cos\beta\lvert x-\xi\rvert)$
$G_3 = -\left(\frac{1}{2}\right)sgn(x-\xi)e^{-\beta\lvert x-\xi\rvert}\cos\beta\lvert x-\xi\rvert$
$H = sgn(x-\xi)\left(\frac{\beta^2}{k}\right)e^{-\beta\lvert x-\xi\rvert}\sin\beta\lvert x-\xi\rvert$
$H_1 = -\left(\frac{\beta^3}{k}\right)e^{-\beta\lvert x-\xi\rvert}(\sin\beta\lvert x-\xi\rvert - \cos\beta\lvert x-\xi\rvert)$
$H_2 = -\left(\frac{1}{2}\right)sgn(x-\xi)e^{-\beta\lvert x-\xi\rvert}\cos\beta\lvert x-\xi\rvert$
$H_3 = -\left(\frac{\beta}{2}\right)e^{-\beta\lvert x-\xi\rvert}[\cos\beta\lvert x-\xi\rvert + \sin\beta\lvert x-\xi\rvert]$

- The equations above are valid for $x > \xi$ and $x < \xi$ but not at $x = \xi$, unless the application of the formula show that the variable is continuous at $x = \xi$.
- During integration it will be necessary to consider the regions $x > \xi$ and $x < \xi$ separately because of the sgn function.

Some properties of fundamental solutions

- $G(x, \xi)$ is an even function about the source point ξ.
- $G_1(x, \xi)$ is an odd function about the source point ξ.
- $G_2(x, \xi)$ is an even function about the source point ξ.
- $G_3(x, \xi)$ is an odd function about the source point ξ.

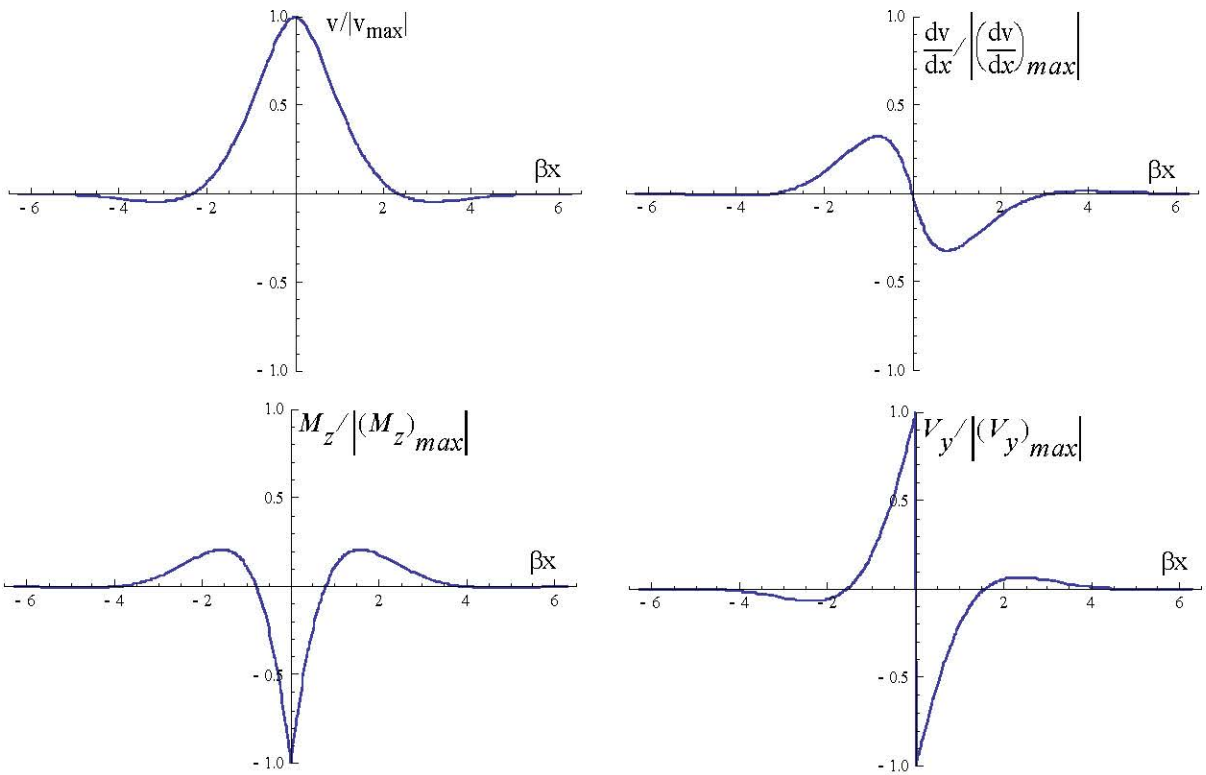
- The displacement is an even function of $(x-\xi)$. The odd derivatives of displacement with respect to x are odd functions and even derivatives are even functions.

$$\mathrm{v}(-x+\xi) = \mathrm{v}(x-\xi) \qquad \frac{\mathrm{dv}}{\mathrm{d}x}(-x+\xi) = \frac{\mathrm{dv}}{\mathrm{d}x}(x-\xi) \qquad M_z(-x+\xi) = M_z(x-\xi) \qquad V_y(-x+\xi) = -V_y(x-\xi)$$

- amplitude decreases exponentially as we move away from the point of application of the force.

$$\left|\mathrm{v}_{max}\right| = P\beta/(2k) \qquad \left|\left(\frac{\mathrm{dv}}{\mathrm{d}x}\right)_{max}\right| = P\beta^2/k \qquad \left|(M_z)_{max}\right| = P/4\beta \qquad \left|(V_y)_{max}\right| = P/2$$

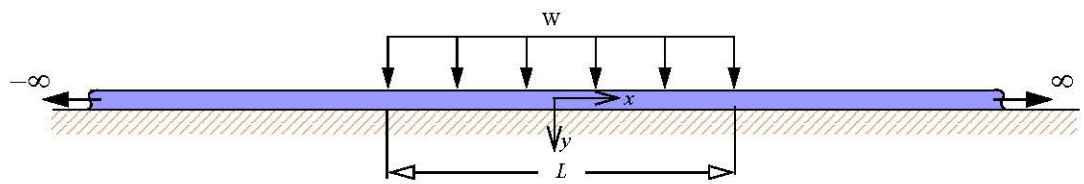
Plot of non-dimensionalized variables in an infinite beam with concentrated force.



- Displacement, slope and moment are continuous at the origin but the shear force jumps by the value of applied load P.

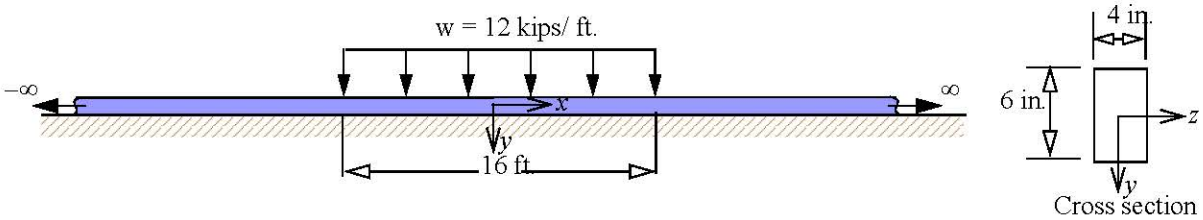
C3.5

Obtain the displacement and internal bending moment function for a point on an infinite beam on elastic foundation under a uniform load as shown



C3.6

A very long rectangular beam with a modulus of elasticity of 30,000 ksi rest on an elastic foundation of modulus of 2 ksi. The beam cross section and loading are as shown below. Determine the maximum deflection, maximum bending normal stress, and the maximum force per unit length acting on the beam.



Non-dimensionalized form of Fundamental Solutions

$$\widehat{\beta} = \beta L_o \qquad \widehat{k} = kL_o/p_o$$

$$\widehat{G} = \left(\frac{1}{8\widehat{\beta}^3}\right) e^{-\widehat{\beta}|\widehat{x}-\widehat{\xi}|}[\cos\widehat{\beta}|\widehat{x}-\widehat{\xi}| + \sin\widehat{\beta}|\widehat{x}-\widehat{\xi}|]$$

$$\widehat{G_1} = -\left(\frac{1}{4\widehat{\beta}^2}\right) sgn(\widehat{x}-\widehat{\xi}) e^{-\widehat{\beta}|\widehat{x}-\widehat{\xi}|} \sin\widehat{\beta}|\widehat{x}-\widehat{\xi}|$$

$$\widehat{G_2} = \left(\frac{1}{4\widehat{\beta}}\right) e^{-\widehat{\beta}|\widehat{x}-\widehat{\xi}|}(\sin\widehat{\beta}|\widehat{x}-\widehat{\xi}| - \cos\widehat{\beta}|\widehat{x}-\widehat{\xi}|)$$

$$\widehat{G_3} = -\left(\frac{1}{2}\right) sgn(\widehat{x}-\widehat{\xi}) e^{-\widehat{\beta}|\widehat{x}-\widehat{\xi}|} \cos\widehat{\beta}|\widehat{x}-\widehat{\xi}|$$

$$\widehat{H} = \left(\frac{1}{4\widehat{\beta}^2}\right) sgn(\widehat{x}-\widehat{\xi}) e^{-\widehat{\beta}|\widehat{x}-\widehat{\xi}|} \sin\widehat{\beta}|\widehat{x}-\widehat{\xi}|$$

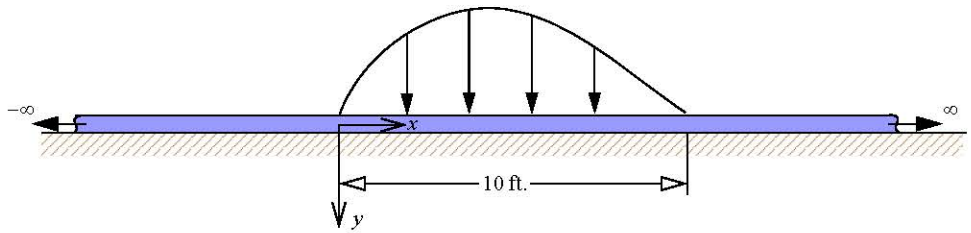
$$\widehat{H_1} = -\left(\frac{1}{4\widehat{\beta}}\right) e^{-\widehat{\beta}|\widehat{x}-\widehat{\xi}|}(\sin\widehat{\beta}|\widehat{x}-\widehat{\xi}| - \cos\widehat{\beta}|\widehat{x}-\widehat{\xi}|)$$

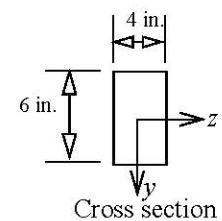
$$\widehat{H_2} = -\left(\frac{1}{2}\right) sgn(\widehat{x}-\widehat{\xi}) e^{-\widehat{\beta}|\widehat{x}-\widehat{\xi}|} \cos\widehat{\beta}|\widehat{x}-\widehat{\xi}|$$

$$\widehat{H_3} = -\left(\frac{\widehat{\beta}}{2}\right) e^{-\widehat{\beta}|\widehat{x}-\widehat{\xi}|}[\cos\widehat{\beta}|\widehat{x}-\widehat{\xi}| + \sin\widehat{\beta}|\widehat{x}-\widehat{\xi}|]$$

C3.7

Figure below shows a very long rectangular beam with a modulus of elasticity of 30,000 ksi that rests on an elastic foundation of modulus of 2 ksi. The beam is subjected to a transverse load of $p_y = 300e^{-(x/10)}\sin(\pi x/10)$ lb/ft. and has the cross section shown. Determine (a) the deflection and bending moment at the mid section of the beam. (b) the maximum deflection and bending normal stress in the beam.





Cross section

$$\hat{x} = x/L_o \qquad \hat{\xi} = \xi/L_o \qquad \hat{p}_y = p_y/p_o \qquad \hat{\beta} = \beta L_o \qquad \hat{k} = kL_o/p_o$$

$$\hat{G} = \left(\frac{1}{8\hat{\beta}^3}\right)e^{-\hat{\beta}|\hat{x}-\hat{\xi}|}[\cos\hat{\beta}|\hat{x}-\hat{\xi}| + \sin\hat{\beta}|\hat{x}-\hat{\xi}|] \qquad \hat{G}_2 = \left(\frac{1}{4\hat{\beta}}\right)e^{-\hat{\beta}|\hat{x}-\hat{\xi}|}(\sin\hat{\beta}|\hat{x}-\hat{\xi}| - \cos\hat{\beta}|\hat{x}-\hat{\xi}|)$$

$$\hat{v}(\hat{x}) = \int_0^1 \hat{G}(\hat{x},\hat{\xi})\hat{p}_y(\hat{\xi})d\hat{\xi} = \left[\sum_{i=1}^{N-1}\hat{G}_i\hat{p}_i + 0.5\{\hat{G}_0\hat{p}_0 + \hat{G}_N\hat{p}_N\}\right](\Delta\hat{\xi})$$

$$\hat{M}_z(\hat{x}) = \int_0^1 \hat{G}_2(\hat{x},\hat{\xi})\hat{p}_y(\hat{\xi})d\hat{\xi} = \left[\sum_{i=1}^{N-1}(\hat{G}_2)_i\hat{p}_i + 0.5\left\{(\hat{G}_2)_0\hat{p}_0 + (\hat{G}_2)_N\hat{p}_N\right\}\right](\Delta\hat{\xi})$$

$L_o = 10$ ft $\qquad (p_y)_{max} = p_o = 191.25$ lb/ft.

$I = (4)(6^3)/12 = 72$ in.^{4} $\qquad EI = 2.160(10^6)$ kips-in.^{2} $= 2.160(10^9)$ lbs-in.^{2}

$$\beta = (k/4EI)^{\frac{1}{4}} = [2/\{4(2.160)(10^6)\}]^{\frac{1}{4}} = 0.021935 \text{ in.}^{-1}$$

$$\hat{\beta} = 0.021935(10)(12) = 2.6321$$

Solution at midpoint using spreadsheet

	A	B	C	D	E	F	G	H	I	J	K
1			0.5								
2	ξ ft.	p_y lb/ft	$\widehat{\xi}$	$\widehat{p}_y$	$\widehat{x}-\widehat{\xi}$	$sgn(\widehat{x}-\widehat{\xi})$	$\widehat{\beta}\lvert\widehat{x}-\widehat{\xi}\rvert$	$\widehat{G}$	$\widehat{G}_i\widehat{p}_i$	$\widehat{G}_2$	$\widehat{G}_{2i}\widehat{p}_i$
3	0	0.00	0	0.0000	0.5	1	1.3161E+00	2.2422E-03	0.0000E+00	1.8232E-02	0.0000E+00
4	1	83.88	0.1	0.4386	0.4	1	1.0529E+00	3.2623E-03	1.4308E-03	1.2387E-02	5.4329E-03
5	2	144.37	0.2	0.7549	0.3	1	7.8964E-01	4.4010E-03	3.3222E-03	2.5895E-04	1.9547E-04
6	3	179.80	0.3	0.9401	0.2	1	5.2643E-01	5.5353E-03	5.2038E-03	-2.0319E-02	-1.9102E-02
7	4	191.25	0.4	1.0000	0.1	1	2.6321E-01	6.4575E-03	6.4575E-03	-5.1491E-02	-5.1491E-02
8	5	181.96	0.5	0.9514	0.0	-1	0.0000E+00	6.8546E-03	6.5214E-03	-9.4979E-02	-9.0364E-02
9	6	156.59	0.6	0.8187	-0.1	-1	2.6321E-01	6.4575E-03	5.2870E-03	-5.1491E-02	-4.2158E-02
10	7	120.52	0.7	0.6302	-0.2	-1	5.2643E-01	5.5353E-03	3.4882E-03	-2.0319E-02	-1.2805E-02
11	8	79.23	0.8	0.4143	-0.3	-1	7.8964E-01	4.4010E-03	1.8233E-03	2.5895E-04	1.0728E-04
12	9	37.69	0.9	0.1971	-0.4	-1	1.0529E+00	3.2623E-03	6.4291E-04	1.2387E-02	2.4411E-03
13	10	0.00	1	0.0000	-0.5	-1	1.3161E+00	2.2422E-03	1.5852E-19	1.8232E-02	1.2890E-18
14									3.4177E-03		-2.0774E-02

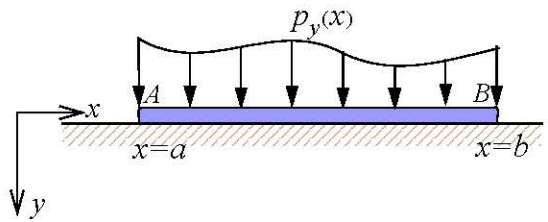
$$\widehat{v} = 3.4177(10^{-3}) \qquad \widehat{M}_z = 20.774(10^{-3})$$

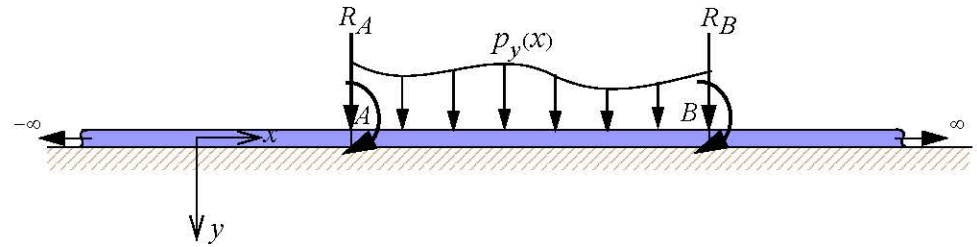
$$v(x=5) = \frac{p_o L^4}{EI}\widehat{v} = \frac{(191.25)(120^4)}{2.160(10^9)}[3.4177(10^{-3})] = 5.229(10^{-3}) \text{ in.}$$

$$M_z(x=5) = p_o L^2 \widehat{M}_z = (191.25)(120^2)(20.774)(10^{-3}) = 47677 \text{ in.-lb}$$

Finite Beams

Consider the finite beam AB as part of an infinite beam with a force and a moment applied in the positive direction at each end. superposition.





$$\mathrm{v}(x) = R_A \mathcal{G}(x,-L/2) + M_A \mathcal{H}(x,-L/2) + R_B \mathcal{G}(x,L/2) + M_B \mathcal{H}(x,L/2) + \int_{-L/2}^{L/2} \mathcal{G}(x,\xi) p_y(\xi) d\xi$$

$$\frac{\partial \mathrm{v}}{\partial x}(x) = R_A \mathcal{G}_1(x,-L/2) + M_A \mathcal{H}_1(x,-L/2) + R_B \mathcal{G}_1(x,L/2) + M_B \mathcal{H}_1(x,L/2) + \int_{-L/2}^{L/2} \mathcal{G}_1(x,\xi) p_y(\xi) d\xi$$

$$M_z(x) = R_A \mathcal{G}_2(x,-L/2) + M_A \mathcal{H}_2(x,-L/2) + R_B \mathcal{G}_2(x,L/2) + M_B \mathcal{H}_2(x,L/2) + \int_{-L/2}^{L/2} \mathcal{G}_2(x,\xi) p_y(\xi) d\xi$$

$$V_y(x) = R_A \mathcal{G}_3(x,-L/2) + M_A \mathcal{H}_3(x,-L/2) + R_B \mathcal{G}_3(x,L/2) + M_B \mathcal{H}_3(x,L/2) + \int_{-L/2}^{L/2} \mathcal{G}_3(x,\xi) p_y(\xi) d\xi$$

Boundary Conditions: $\boxed{\mathrm{v} \quad or \quad V_y}$ and $\boxed{\frac{\mathrm{dv}}{\mathrm{d}x} \quad or \quad M_z}$

Four equations in four unknowns R_A M_A R_B M_B

Left boundary at A is considered then $x = -L/2 + \varepsilon$

Right boundary at B is considered then $x = L/2 - \varepsilon$.

We then let $\varepsilon \to 0$ to get the correct signs for the *sgn* function.

Symmetric loading and boundary conditions

We assume that the beam has symmetric boundary conditions and loading, that is $p_y(x) = p_y(-x) = p_s(x)$

$$R_B = R_A = R_s \qquad \text{and} \qquad M_B = -M_A = -M_s$$

$$R_s[\mathcal{G}(x,-L/2) + \mathcal{G}(x,L/2)] + M_s[\mathcal{H}(x,-L/2) - \mathcal{H}(x,L/2)] + \int_{-L/2}^{L/2} \mathcal{G}(x,\xi) \tag{3.1a}$$

$$R_s[\mathcal{G}_1(x,-L/2) + \mathcal{G}_1(x,L/2)] + M_s[\mathcal{H}_1(x,-L/2) - \mathcal{H}_1(x,L/2)] + \int_{-L/2}^{L/2} \mathcal{G}_1(x,\xi) \tag{3.1b}$$

$$R_s[\mathcal{G}_2(x,-L/2) + \mathcal{G}_2(x,L/2)] + M_s[\mathcal{H}_2(x,-L/2) - \mathcal{H}_2(x,L/2)] + \int_{-L/2}^{L/2} \mathcal{G}_2(x,\xi) \tag{3.1c}$$

$$R_s[\mathcal{G}_3(x,-L/2) + \mathcal{G}_3(x,L/2)] + M_s[\mathcal{H}_3(x,-L/2) - \mathcal{H}_3(x,L/2)] + \int_{-L/2}^{L/2} \mathcal{G}_3(x,\xi) \tag{3.1d}$$

Need two boundary conditions at one of the ends to obtain R_s and M_s.

Asymmetric loading and boundary conditions

We assume that the beam has asymmetric boundary conditions and the loading, that is, $p_y(x) = -p_y(-x) = p_a(x)$.

$$R_B = -R_A = -R_a \qquad \text{and} \qquad M_B = M_A = M_a$$

$$\mathrm{v}(x) = R_a[\mathcal{G}(x,-L/2) - \mathcal{G}(x,L/2)] + M_a[\mathcal{H}(x,-L/2) + \mathcal{H}(x,L/2)] + \int_{-L/2}^{L/2} \mathcal{G}(x,\xi)p_a(x)d\xi$$

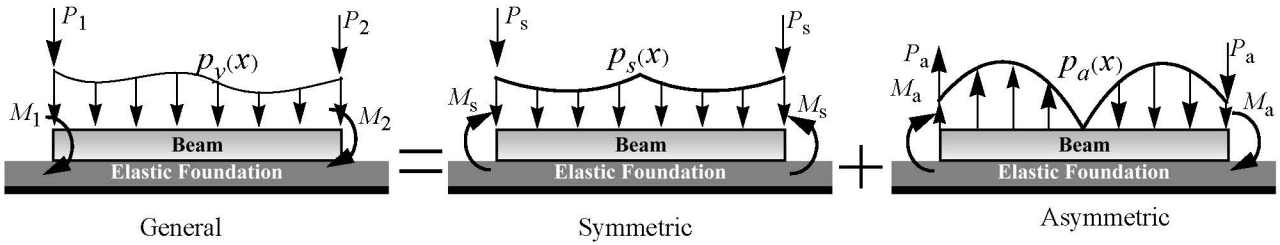
$$\frac{\partial \mathrm{v}}{\partial x}(x) = R_a[\mathcal{G}_1(x,-L/2) - \mathcal{G}_1(x,L/2)] + M_a[\mathcal{H}_1(x,-L/2) + \mathcal{H}_1(x,L/2)] + \int_{-L/2}^{L/2} \mathcal{G}_1(x,\xi)p_a(x)d\xi$$

$$M_z(x) = R_a[\mathcal{G}_2(x,-L/2) - \mathcal{G}_2(x,L/2)] + M_a[\mathcal{H}_2(x,-L/2) + \mathcal{H}_2(x,L/2)] + \int_{-L/2}^{L/2} \mathcal{G}_2(x,\xi)p_a(x)d\xi$$

$$V_y(x) = R_a[\mathcal{G}_3(x,-L/2) - \mathcal{G}_3(x,L/2)] + M_a[\mathcal{H}_3(x,-L/2) + \mathcal{H}_3(x,L/2)] + \int_{-L/2}^{L/2} \mathcal{G}_3(x,\xi)p_a(x)d\xi$$

General loading and boundary conditions

- Any function can be written as a sum of a symmetric function and asymmetric function



$$p_s(x) = [p_y(x) + p_y(-x)]/2 \qquad P_s = [P_1 + P_2]/2 \qquad M_s = [M_1 + M_2]/2$$

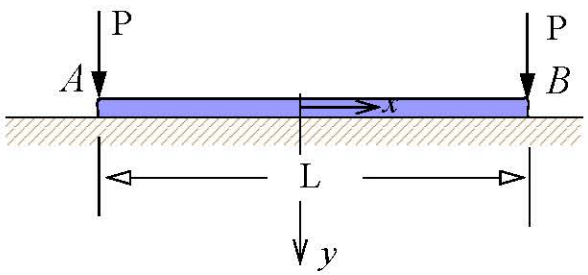
$$p_a(x) = [p_y(x) - p_y(-x)]/2 \qquad P_a = [P_1 - P_2]/2 \qquad M_a = [M_1 - M_2]/2$$

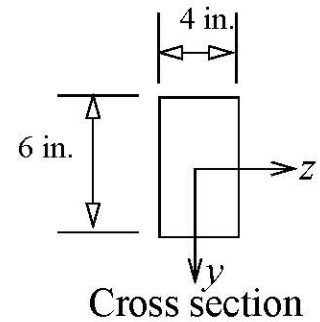
$$P_1 = (P_1 + P_2)/2 + (P_1 - P_2)/2 = P_s + P_a \quad \text{and} \quad P_2 = (P_1 + P_2)/2 - (P_1 - P_2)/2 = P_s - P_a$$

$$R_A = R_s + R_a \qquad R_B = R_s - R_a \qquad M_A = M_s + M_a \qquad M_B = -M_s + M_a$$

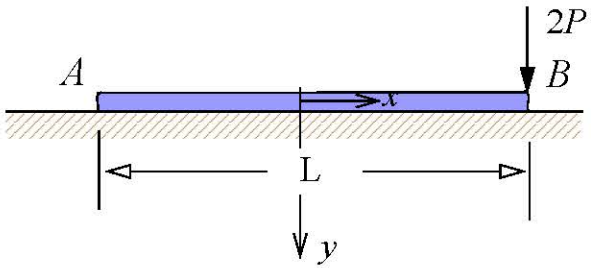
C3.8

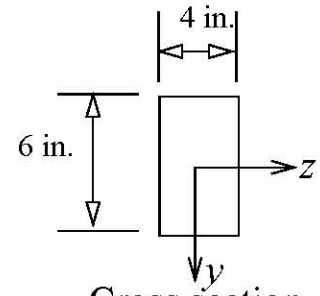
A finite beam on elastic foundation is loaded as shown in figure below. (a) In terms of E, I, b, k, and L determine the unknown force R_s and moment M_s. (b) Make a plot of the deflection, slope, internal bending moment, and internal shear force across the beam assuming $k = 2\text{ ksi}$, $\beta = 0.021935\text{ in.}^{-1}$, $P = 1.5\text{ kips}$ and $L = 10\text{ ft}$.





C3.9 A finite beam on elastic foundation is loaded as shown below. (a) In terms of E, I, β, k, and L determine the unknown forces R_A, R_B and moments M_A, M_B. (b) Make a plot of the deflection, slope, internal bending moment, and internal shear force across the beam assuming the beam cross-section and material properties of the beam and foundation are the same as in previous problem.





Cross section

$$I = (4)(6^3)/12 = 72 \text{ in.}^4 \qquad EI = 2.160(10^6) \text{ kips-in.}^2 = 2.160(10^9) \text{ lbs-in.}^2$$

$$\beta = (k/4EI)^{\frac{1}{4}} = [2/\{4(2.160)(10^6)\}]^{\frac{1}{4}} = 0.021935 \text{ in.}^{-1} = 0.26321 \text{ ft.}^{-1} \qquad \beta L = 2.6321 \qquad P = 1.5 \text{ kips}$$

$$R_A = 5.2801 \text{ kips} \qquad M_A = -9.6547 \text{ ft-kips}$$

$$\mathrm{v}(x) = R_A[G(x,-L/2) + G(x,L/2)] + M_A[H(x,-L/2) - H(x,L/2)]$$

$$\frac{\partial \mathrm{v}}{\partial x}(x) = R_A[G_1(x,-L/2) + G_1(x,L/2)] + M_A[H_1(x,-L/2) - H_1(x,L/2)]$$

$$\boldsymbol{M}_z(x) = R_A[G_2(x,-L/2) + G_2(x,L/2)] + M_A[H_2(x,-L/2) - H_2(x,L/2)]$$

$$\boldsymbol{V}_y(x) = R_A[G_3(x,-L/2) + G_3(x,L/2)] + M_A[H_3(x,-L/2) - H_3(x,L/2)]$$

Maximum values and the location.

	v_{max}	$\left(\frac{\mathrm{dv}}{\mathrm{d}x}\right)_{max}$	$(\boldsymbol{M}_z)_{max}$	$(\boldsymbol{V}_y)_{max}$
Location x	±5 ft.	±5 ft.	0	±5 ft.
Value	2.264(10^{-3}) ft.	0.7160(10^{-3}) rads.	2.5778 ft.-kips	∓1.5 kips