One-Dimensional Structural Members

(a) (b) (c)

Courtesy (a) Jaksmata http://en.wikipedia.org/wiki/File:Wood-framed_house.jpg (b) Sklmata http://en.wikipedia.org/wiki/File:Human-Skeleton.jpg (c) jrok http://en.wikipedia.org/wiki/File:ToyotaTundraChassis.jpg

The learning objectives in this chapter are:

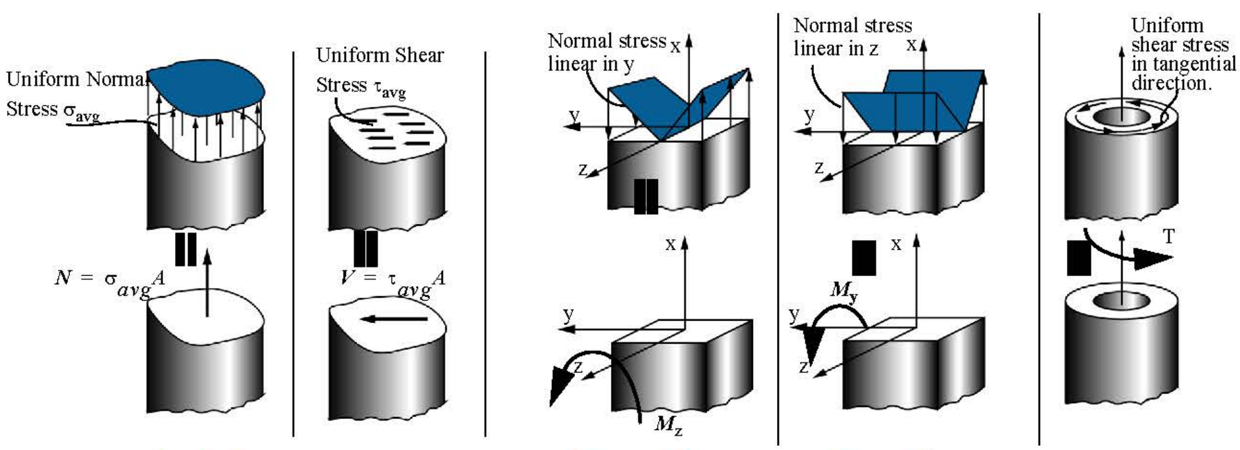
- Understand the limitations of basic theory and how complexities may be added to the basic theories of axial members, torsion of circular shafts, and symmetric bending of beams.

- Understand the concept and use of discontinuity functions in analysis of structural members subjected to discontinuous loads.

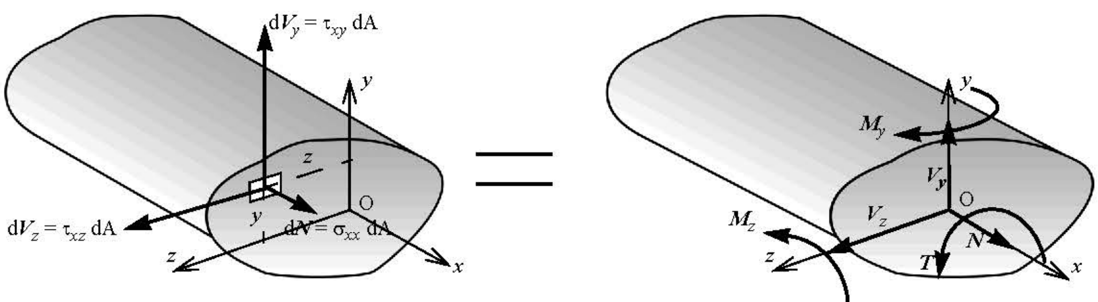
Internal Forces (Stress Resultants)

- Stress components are internal force distribution that act on a surface. In Statics we learned that any distributed force can be replaced by an equivalent force and moment at any point. It is this principle of static equivalency that we use to relate stresses to internal forces and moments.
- Relating stresses to external forces and moments is a two step process.

Static equivalency



Axial **Bending** **Bending** **Torsion**

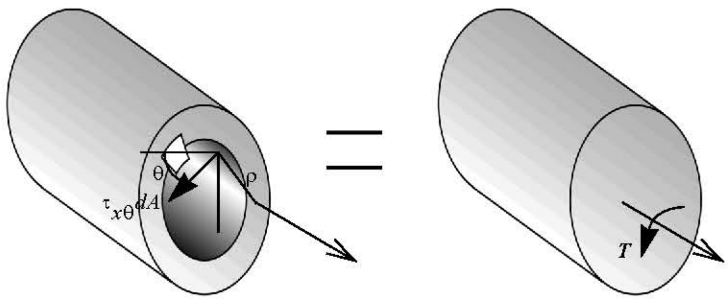


(a) (b)

$$N = \int_A \sigma_{xx} dA \qquad V_y = \int_A \tau_{xy} dA \qquad V_z = \int_A \tau_{xz} dA$$

$$M_y = -\int_A z\sigma_{xx} dA \qquad M_z = -\int_A y\sigma_{xx} dA \qquad T = \int_A [y dV_z - z dV_y] = \int_A [y\tau_{xz} - z\tau_{xy}] dA$$

- Internal forces and moments are positive and negative according to a sign convention that is derived from sign convention for stresses.
- There are specific points in space which decouples the normal stress due to bending from that due to axial, and the shear stress due to bending from that due to torsion.



For circular cross-sections with only shear stress $\tau_{x\theta}$, the shear forces can be written as $dV_z = (\tau_{xz} dA) = (\tau_{x\theta} dA)\cos\theta$; $dV_y = (\tau_{xy} dA) = -(\tau_{x\theta} dA)\sin\theta$ and $y = \rho\cos\theta$; $z = \rho\sin\theta$. Substituting these into torque expression we obtain the torque on circular shafts as:

$$T = \int_A [(\rho\cos\theta)(\tau_{x\theta}\cos\theta) - (\rho\sin\theta)(-\tau_{x\theta}\sin\theta)] dA \text{ or}$$

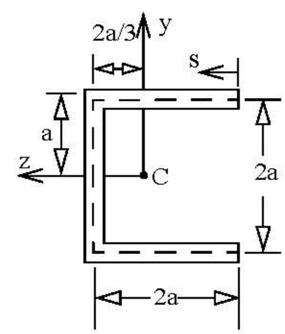
$$T = \int_A \rho\tau_{x\theta}[\cos^2\theta + \sin^2\theta] dA = \int_A \rho\tau_{x\theta} dA$$

C2.1 The cross-section shown has a uniform thickness t. Assuming $t \ll a$ the shear stresses in the cross section were found and are as given. (a) Replace the shear stresses by equivalent shear forces and torque acting at the centroid C. (b) Determine the location of the point where the shear stresses can be replaced by just shear forces and no torque. [Such points are called shear centers.]

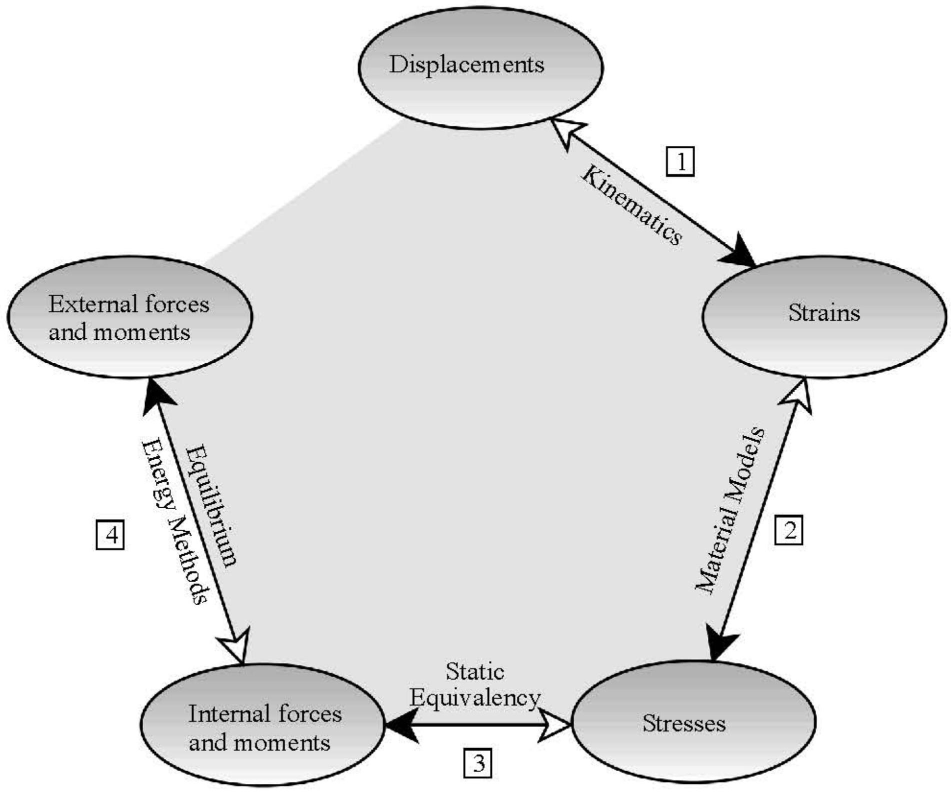
$$\tau_{xy} = 0 \qquad \tau_{xz} = Ks/t \qquad 0 \le s < 2a$$

$$\tau_{xy} = -K(-4a^2 + 6as - s^2)/(2at) \qquad \tau_{xz} = 0 \qquad 2a < s < 4a$$

$$\tau_{xy} = 0 \qquad \tau_{xz} = K(s-6a)/t \qquad 4a < s \le 6a$$



Logic in structural analysis

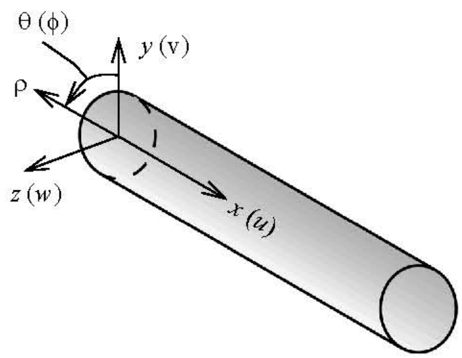


Preliminaries

Limitations

- The length of the member is significantly greater (approximately 10 times) then the greatest dimension in the cross-section. Approximation across the cross-section are now possible as the region of approximation is small.
- We are away from regions of stress concentration, where displacements and stresses can be three-dimensional.
- The variation of external loads or changes in the cross-sectional area is gradual except in regions of stress concentration.
- The external loads are such that the axial, torsion and bending problems can be studied individually.

Convention



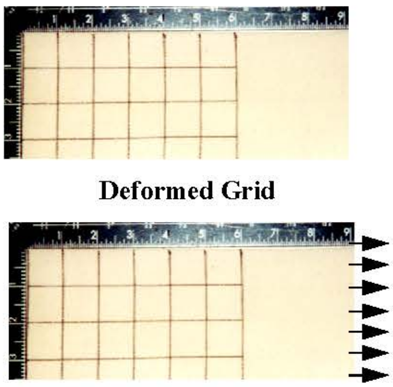
- The displacements u, v, and w will be considered to be positive in the positive x, y, and z direction, respectively.
- The rotation ϕ of the cross section will be considered to be positive counter clockwise with respect to the x axis.
- The external distributed torque per unit length $t(x)$ is positive counterclockwise with respect to the x axis.
- The external distributed forces per unit length $p_x(x)$ and $p_y(x)$ are considered to be positive in the positive x and y direction, respectively.

Deformations

(a) **Axial**

Original Grid

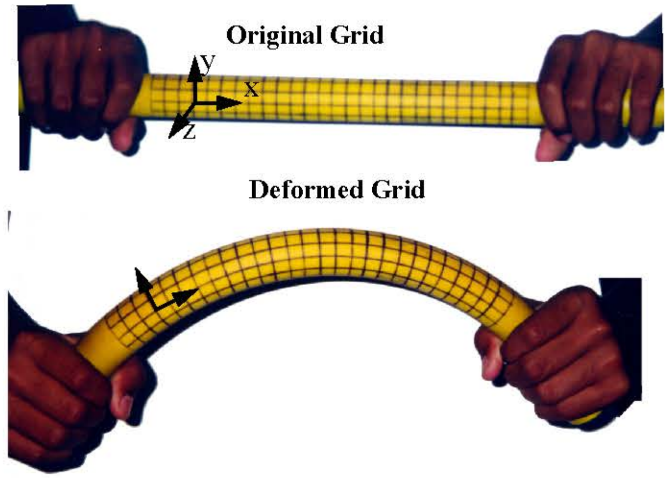
Deformed Grid



(b) **Bending**

Original Grid

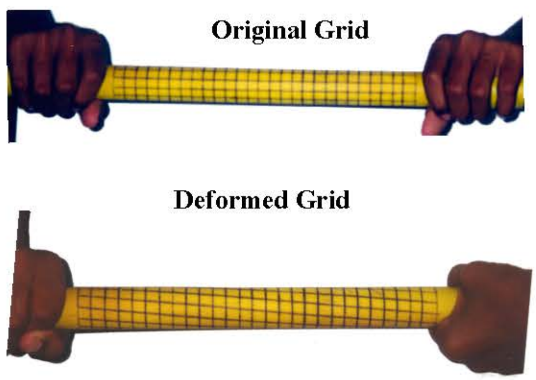
Deformed Grid



(c) **Torsion**

Original Grid

Deformed Grid



	Axial	Bending	Torsion
Assumption 1 Deformations are not function of time.			
Assumptions	2-A: Plane sections remain plane and parallel.	2a-B: Squashing deformation is significantly smaller than deformation due to bending. 2b-B: Plane sections before deformation remain plane after deformation. 2c-B: Plane perpendicular to the beam axis remain *nearly* perpendicular after deformation	2a-T: Plane sections perpendicular to the axis remain plane during deformation. 2b-T: All radials lines rotate by equal angle during deformation on a cross-section. 2c-T: Radials lines remain straight during deformation.
	$u = u_o(x)$ **(2.5-A)**	$v = v(x)$ **(2.5a-B)** $u = -y\frac{dv}{dx}$ **(2.5b-B)**	$\phi = \phi(x)$ **(2.5-T)**

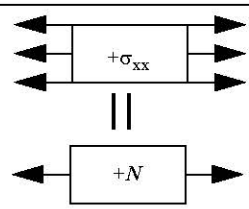
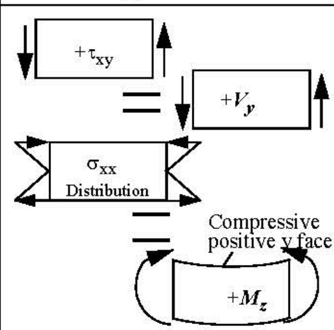
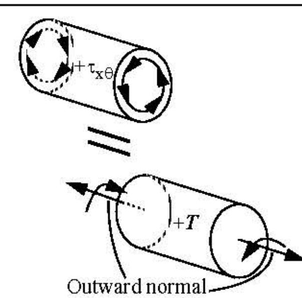
Strains

	Axial	Bending	Torsion
Assumption 3 The strains are small.			
	$\varepsilon_{xx} = \frac{du_o}{dx}(x)$ **(2.6-A)**	$\varepsilon_{xx} = -y\frac{d^2v}{dx^2}(x)$ **(2.6-B)**	$\gamma_{x\theta} = \rho\frac{d\phi}{dx}(x)$ **(2.6-T)**

Stresses

	Axial	Bending	Torsion
Assumption 4 Material is isotropic. Assumption 5 There are no inelastic strains. Assumption 6 Material is elastic. Assumption 7 Stress and strains are linearly related.			
Using Hooke's law	$\sigma_{xx} = E\frac{du_o}{dx}(x)$ **(2.7-A)**	$\sigma_{xx} = -Ey\frac{d^2v}{dx^2}(x)$ **(2.7-B)**	$\tau_{x\theta} = G\rho\frac{d\phi}{dx}(x)$ **(2.7-T)**

Internal Forces and Moments

	Axial	Bending	Torsion
Static equivalency	$N = \int_A \sigma_{xx} dA$ **(2.8a-A)** $M_z = -\int_A y\sigma_{xx} dA = 0$ **(2.8b-A)** $M_y = -\int_A z\sigma_{xx} dA = 0$ **(2.8c-A)**	$N = \int_A \sigma_{xx} dA = 0$ **(2.8a-B)** $M_z = -\int_A y\sigma_{xx} dA$ **(2.8b-B)** $V_y = \int_A \tau_{xy} dA$ **(2.8c-B)**	$T = \int_A \rho\tau_{x\theta} dA$ **(2.8-T)**
Sign convention			

Formulas

	Axial	Bending	Torsion
Origin Location	$\int_A yE\,dA = 0$ **(2.9-A)**	$\int_A yE\,dA = 0$ **(2.9-B)**	
	$N = \frac{du_o}{dx}\int_A E\,dA$ **(2.10-A)**	$M_z = \frac{d^2v}{dx^2}\int_A Ey^2\,dA$ **(2.10-B)**	$T = \frac{d\phi}{dx}\int_A G\rho^2\,dA$ **(2.10-T)**
Assumption 8 Material is homogenous across the cross-section.			
Origin is at the centroid of the cross-section	$\int_A y\,dA = 0$ **(2.11-A)**	$\int_A y\,dA = 0$ **(2.11-B)**	
	$\frac{du_o}{dx} = \frac{N}{EA}$ **(2.12-A)**	$\frac{d^2v}{dx^2} = \frac{M_z}{EI_{zz}}$ **(2.12-B)**	$\frac{d\phi}{dx} = \frac{T}{GJ}$ **(2.12-T)**
	A = Area of cross-section EA = Axial Rigidity	I_{zz} = Second area moment of inertia EI_{zz} = Bending rigidity	J= Polar moment of the area. GJ = Torsional rigidity

Stress formulas

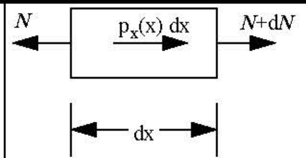
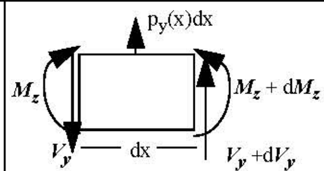
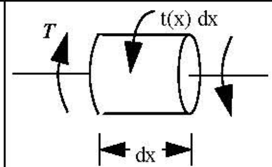
Substituting Equations (2.12-A), (2.12-B), and (2.12-T) into Equations (2.7-A), (2.7-B), and (2.7-T)

	Axial	Bending	Torsion
	$\sigma_{xx} = \frac{N}{A}$ **(2.13-A)**	$\sigma_{xx} = -\left(\frac{M_z y}{I_{zz}}\right)$ **(2.13-B)** Shear stress	$\tau_{x\theta} = \frac{T\rho}{J}$ **(2.13-T)**

Deformation formulas

	Axial	Bending	Torsion
Assumption 9 Material is homogenous between x_1 and x_2. Assumption 10 The structural member is not tapered between x_1 and x_2. Assumption 11 The external loads do not change with x between x_1 and x_2. Integrating Equations (2.12-A) and (2.12-T)			
	$u_2 - u_1 = \frac{N(x_2 - x_1)}{EA}$ **(2.14-A)**	See Section 3.2.4 for beam deflection.	$\phi_2 - \phi_1 = \frac{T(x_2 - x_1)}{GJ}$ **(2.14-T)**

Equilibrium Equations

	Axial	Bending	Torsion
	 $\frac{dN}{dx} = -p_x(x)$ **(2.15-A)**	 $\frac{dV_y}{dx} = -p_y(x)$ **(2.15a-B)** $\frac{dM_z}{dx} = -V_y$ **(2.15b-B)**	 $\frac{dT}{dx} = -t(x)$ **(2.15-T)**

Differential Equations

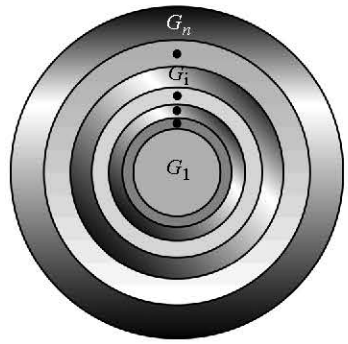
Substituting Equations (2.12-A), (2.12-B), and (2.12-T) into Equations (2.15-A), (2.15a-B), (2.15b-B), and (2.15-T)

	$\frac{d}{dx}\left(EA\frac{du_o}{dx}\right) = -p_x(x)$ **(2.16-A)**	$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = p_y(x)$ **(2.16-B)**	$\frac{d}{dx}\left(GJ\frac{d\phi}{dx}\right) = -t(x)$ **(2.16-T)**

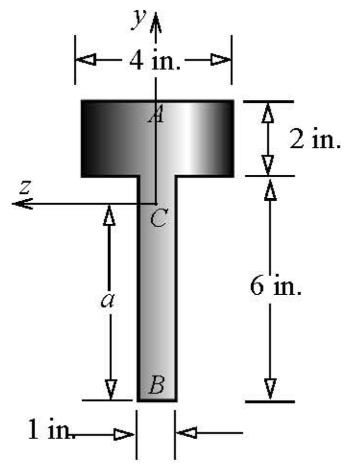
C2.2 Figure below shows a laminated shaft in which all materials are securely bonded together. All assumptions except homogeneity across the cross-section are valid. Show the equation relating torsional shear stress in the i$^{\text{th}}$ material $(\tau_{x\theta})_i$ and internal torque $\boldsymbol{T}$ and the differential equation governing the shaft rotation $\phi(x)$ are as shown below.

$$(\tau_{x\theta})_i = \frac{G_i \rho \boldsymbol{T}}{\sum_{j=1}^{n} G_j J_j}$$

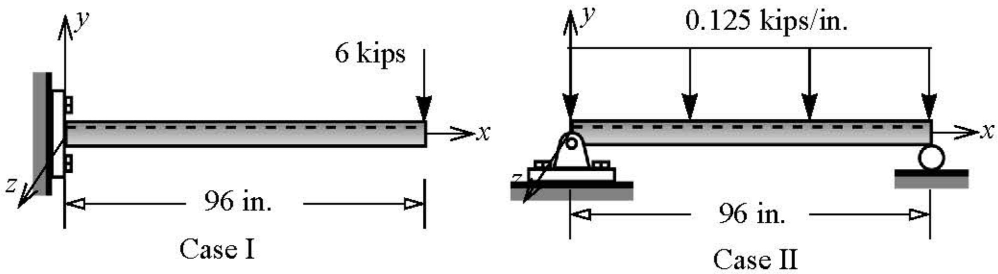
$$\frac{d}{dx}\left[\left\{\sum_{j=1}^{n} G_j J_j\right\}\frac{d\phi}{dx}\right] = -t(x)$$



C2.3 The beam cross section shown is made from a material that has a stress–strain curve given by $\sigma = 400\varepsilon^{0.4}$ ksi. Determine (a) the location of the neutral axis and (b) the bending normal stress in terms of y and the internal moment M_z.



C2.4 Determine the maximum bending normal stress for the two beams and loading shown in the figure below. The beam cross section and material are the same as in problem C2.3.



C2.5 In Timoshenko beams the assumption of planes remaining perpendicular to the axis of the beam is dropped to account for shear by permitting the cross section to rotate by an angle ψ from the vertical. Obtain the differential equations for vibration of Timoshenko beam by starting with the following displacement field

$$u = -y\psi(x, t) \qquad \mathrm{v} = \mathrm{v}(x, t)$$

Boundary Value Problems

Axial Displacement

Strain: $\frac{\mathrm{d}u_0}{\mathrm{d}x} = \frac{\boldsymbol{N}}{EA}$

Equilibrium equations $\frac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}x} = -p_x(x)$

Differential equation: $\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u_0}{\mathrm{d}x}\right) = -p_x(x)$

Boundary conditions u_0 or $\boldsymbol{N}$

Solution: $EA\frac{du_0}{dx} = I_1(x) + C_1 \qquad I_1(x) = -\int p_x(x)dx$

If EA is a constant: $EAu_0 = I_2(x) + C_1x + C_2 \qquad I_2(x) = \int I_1(x)dx$

Homogeneous solution: $u_H = (C_1x + C_2)/EA$

Particular solution: $u_P = I_2(x)/EA$

Loading Integrals I_1, I_2

Torsional Rotation

Rate of rotation: $\frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\boldsymbol{T}}{GJ}$

Equilibrium equation: $\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}x} = -t(x)$

Differential equation: $\frac{\mathrm{d}}{\mathrm{d}x}\left(GJ\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) = -t(x)$

Boundary conditions ϕ or $\boldsymbol{T}$

Solution: $GJ\frac{d\phi}{dx} = I_1(x) + C_1 \qquad I_1(x) = -\int t(x)dx$

If GJ is a constant: $GJ\phi = I_2(x) + C_1x + C_2 \qquad I_2(x) = \int I_1(x)dx$

Homogeneous solution: $\phi_H = (C_1x + C_2)/GJ$

Particular solution: $\phi_P = I_2(x)/GJ$

Loading Integrals I_1, I_2

Beam Deflection

Moment curvature: $M_z = EI_{zz}\dfrac{d^2v}{dx^2}$

Equilibrium equations: $\dfrac{dV_y}{dx} = -p_y(x) \qquad V_y = -\left(\dfrac{dM_z}{dx}\right)$

Shear Force: $V_y = -\left[\dfrac{d}{dx}\left(EI_{zz}\dfrac{d^2v}{dx^2}\right)\right]$

Fourth order differential equation: $\dfrac{d^2}{dx^2}\left(EI_{zz}\dfrac{d^2v}{dx^2}\right) = p_y(x)$

Boundary conditions *Group* 1 v or V_y

and

Group 2 $\dfrac{dv}{dx}$ or M_z

Solution:

$$\frac{d}{dx}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = I_1(x) + C_1 \qquad I_1(x) = \int p_y(x)dx$$

$$EI_{zz}\frac{d^2v}{dx^2} = I_2(x) + C_1x + C_2 \qquad I_2(x) = \int I_1(x)dx$$

If EI_{zz} is a constant:

$$EI_{zz}\frac{dv}{dx} = I_3(x) + C_1(x^2/2) + C_2x + C_3$$

$$I_3(x) = \int I_2(x)dx$$

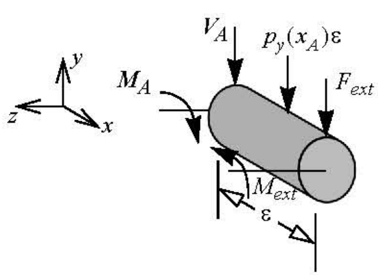
$$EI_{zz}v = I_4(x) + C_1(x^3/6) + C_2(x^2/2) + C_3x + C_4$$

$$I_4(x) = \int I_3(x)dx$$

Homogeneous solution: $v_H = (C_1(x^3/6) + C_2(x^2/2) + C_3x + C_4)/(EI_{zz})$

Particular solution: $v_P = I_4(x)/(EI_{zz})$

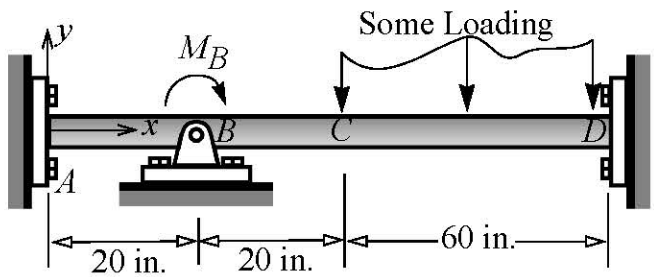
Loading Integrals I_1, I_2, I_3, I_4



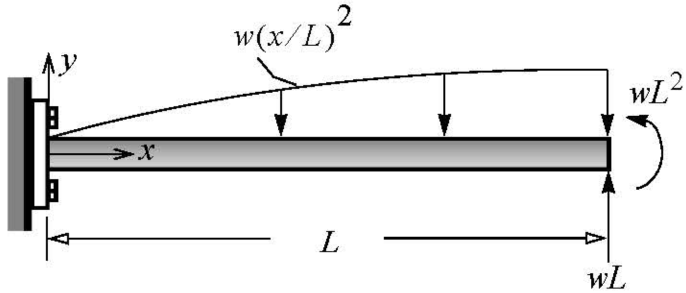
$$\lim_{\varepsilon \to 0}(F_{\text{ext}} + V_A - p_y(x_A)\varepsilon) = 0 \qquad \text{or} \qquad V_A = -F_{\text{ext}}$$

$$\lim_{\varepsilon \to 0}\left(M_A - M_{\text{ext}} + \varepsilon F_{\text{ext}} + p_y(x_A)\frac{\varepsilon^2}{2}\right) = 0 \qquad \text{or} \qquad M_A = M_{\text{ext}}$$

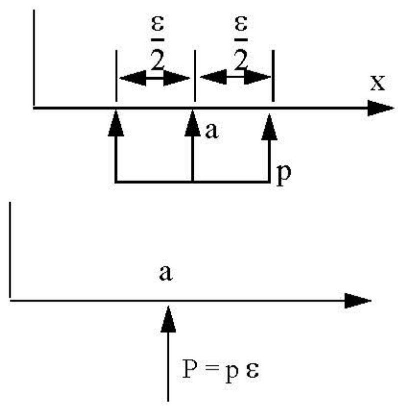
C2.6 The displacement of the beam in the y-direction, in section *AB* of the beam shown is given by = $5(x^3 - 20x^2)\ (10^{-6})$ and in section *BC* is given : $5(x^3 - 800x + 8000)\ (10^{-6}$. If the bending rigidity (EI) is 135 (10^6) lbs-in_2, determine the moment M_B and the reaction force at B.

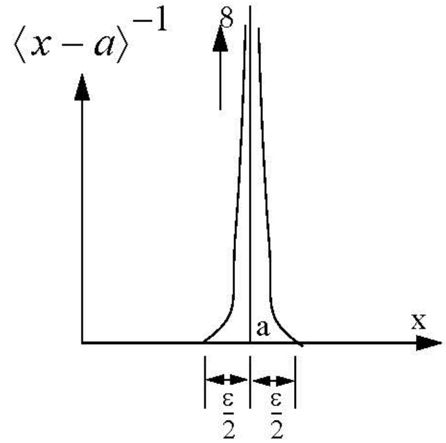


C2.7 In terms of w, L, E, and I, determine the deflection and slope at $x = L$ of the beam shown.



Discontinuity Functions



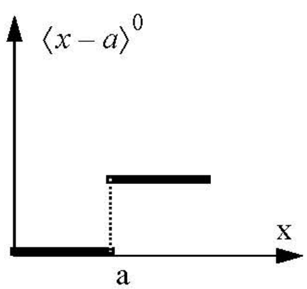


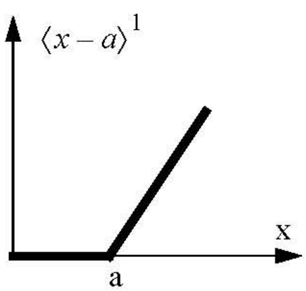
$$P = \lim_{p \to \infty} \lim_{\varepsilon \to 0} (p\varepsilon) \quad \text{or} \quad \langle x-a \rangle^{-1} = \begin{Bmatrix} 0 & x \neq a \\ \infty & x \to a \end{Bmatrix} \qquad \int_{(a-\varepsilon)}^{(a+\varepsilon)} \langle x-a \rangle^{-1} dx = 1$$

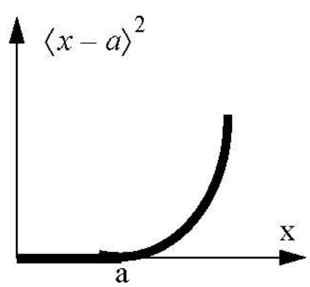
Delta Function: $\langle x-a \rangle^{-1}$

$$\int_{-\infty}^{x} \langle x-a \rangle^{-1} dx = \int_{-\infty}^{(a-\varepsilon)} \langle x-a \rangle^{-1} dx + \int_{(a-\varepsilon)}^{(a+\varepsilon)} \langle x-a \rangle^{-1} dx + \int_{(a+\varepsilon)}^{x} \langle x-a \rangle^{-1} dx = 1$$

$$\langle x-a \rangle^{0} = \int_{-\infty}^{x} \langle x-a \rangle^{-1} dx = \begin{Bmatrix} 0 & x < a \\ 1 & x > a \end{Bmatrix}$$







$$\langle x-a \rangle^{n} = \begin{cases} 0 & x \le a \\ (x-a)^n & x > a \end{cases}$$

$$\int_{-\infty}^{x} \langle x-a \rangle^{n} dx = \frac{\langle x-a \rangle^{n+1}}{(n+1)} \qquad n \ge 0$$

Doublet Function: $\langle x-a\rangle^{-2} = \begin{cases} 0 & x \neq a \\ \infty & x \to a \end{cases}$ $\quad \int_{-\infty}^{x} \langle x-a\rangle^{-2} dx = \langle x-a\rangle^{-1}$

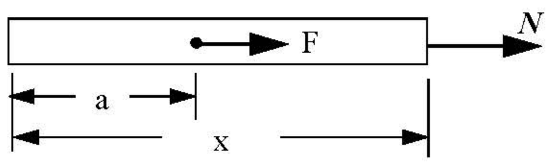
$$\frac{\mathrm{d}\langle x-a\rangle^{-1}}{\mathrm{d}x} = \langle x-a\rangle^{-2} \qquad \frac{\mathrm{d}\langle x-a\rangle^{0}}{\mathrm{d}x} = \langle x-a\rangle^{-1}$$

$$\frac{\mathrm{d}\langle x-a\rangle^{n}}{\mathrm{d}x} = n\langle x-a\rangle^{n-1} \qquad n \geq 1$$

- The function delta function $\langle x-a\rangle^{-1}$ and the doublet function $\langle x-a\rangle^{-2}$ become infinite at x = a. Alternatively stated these functions are singular at x = a. and are referred to as *singularity functions*.
- The entire class of functions $\langle x-a\rangle^{n}$ for positive and negative ‘n’ are called the *discontinuity functions*.

Templates

Axial Displacement

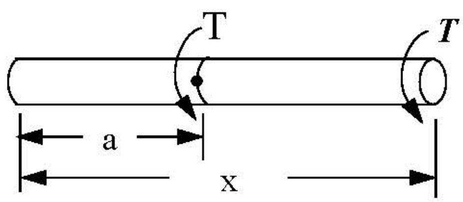


Template equations

$$N = -F\langle x-a\rangle^{0}$$

$$p_x = F\langle x-a\rangle^{-1}$$

Torsional Rotation

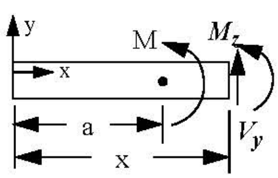


Template equations

$$T = -T\langle x-a\rangle^{0}$$

$$t = T\langle x-a\rangle^{-1}$$

Beam Deflection

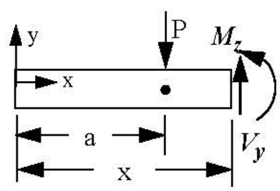


$$M_z = \begin{cases} 0 & x < a \\ -M & x > a \end{cases}$$

Template equations

$$M_z = -M\langle x-a\rangle^{0}$$

$$p_y = -M\langle x-a\rangle^{-2}$$

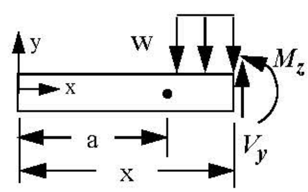


$$M_z = \begin{cases} 0 & x < a \\ -P(x-a) & x > a \end{cases}$$

Template equations

$$M_z = -P\langle x-a\rangle^{1}$$

$$p_y = -P\langle x-a\rangle^{-1}$$



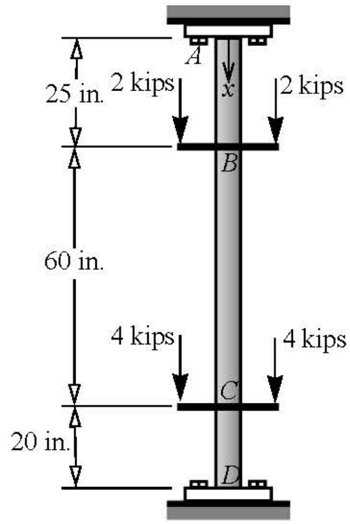
$$M_z = \begin{cases} 0 & x < a \\ -\dfrac{w(x-a)^2}{2} & x > a \end{cases}$$

Template equations

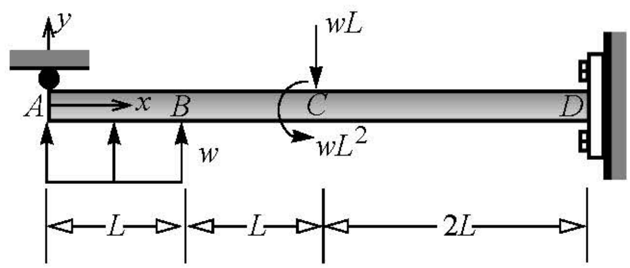
$$M_z = -w\frac{\langle x-a\rangle^{2}}{2}$$

$$p_y = -w\langle x-a\rangle^{0}$$

C2.8 The column shown has a specific weight of $\gamma = 0.1\ \text{lb/in}^3$, modulus of elasticity of E = 4000 ksi and area of cross-section of $A = 100\ \text{in}^2$. (a) Determine the movement of rigid plate at C. (b) The reaction force at A.



C2.9 (a) Determine the deflection of the beam at point C in terms of E, I, w, and L for the beam shown.(b) Determine the maximum bending moment and shear force.



Class Problem 3.1

For the beam and loading shown write the boundary value problem. Assume the bending rigidity *EI* is a constant. DO NOT SOLVE.