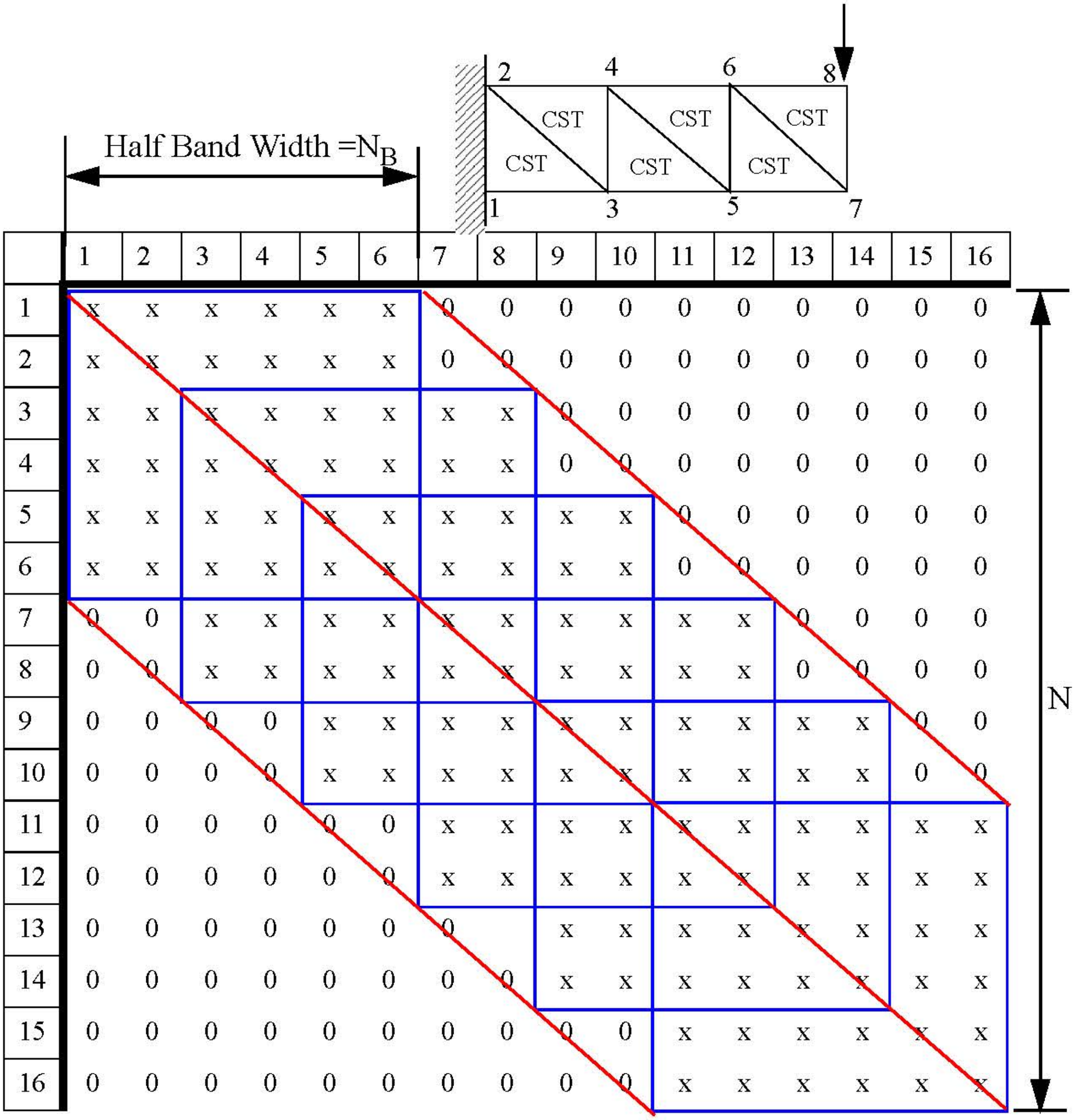
Storage and Solution Techniques

Banded Matrix

- Each node has two degrees of freedom (u and v)
- Element matrix is 6 x 6



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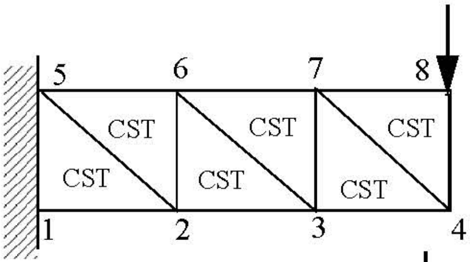
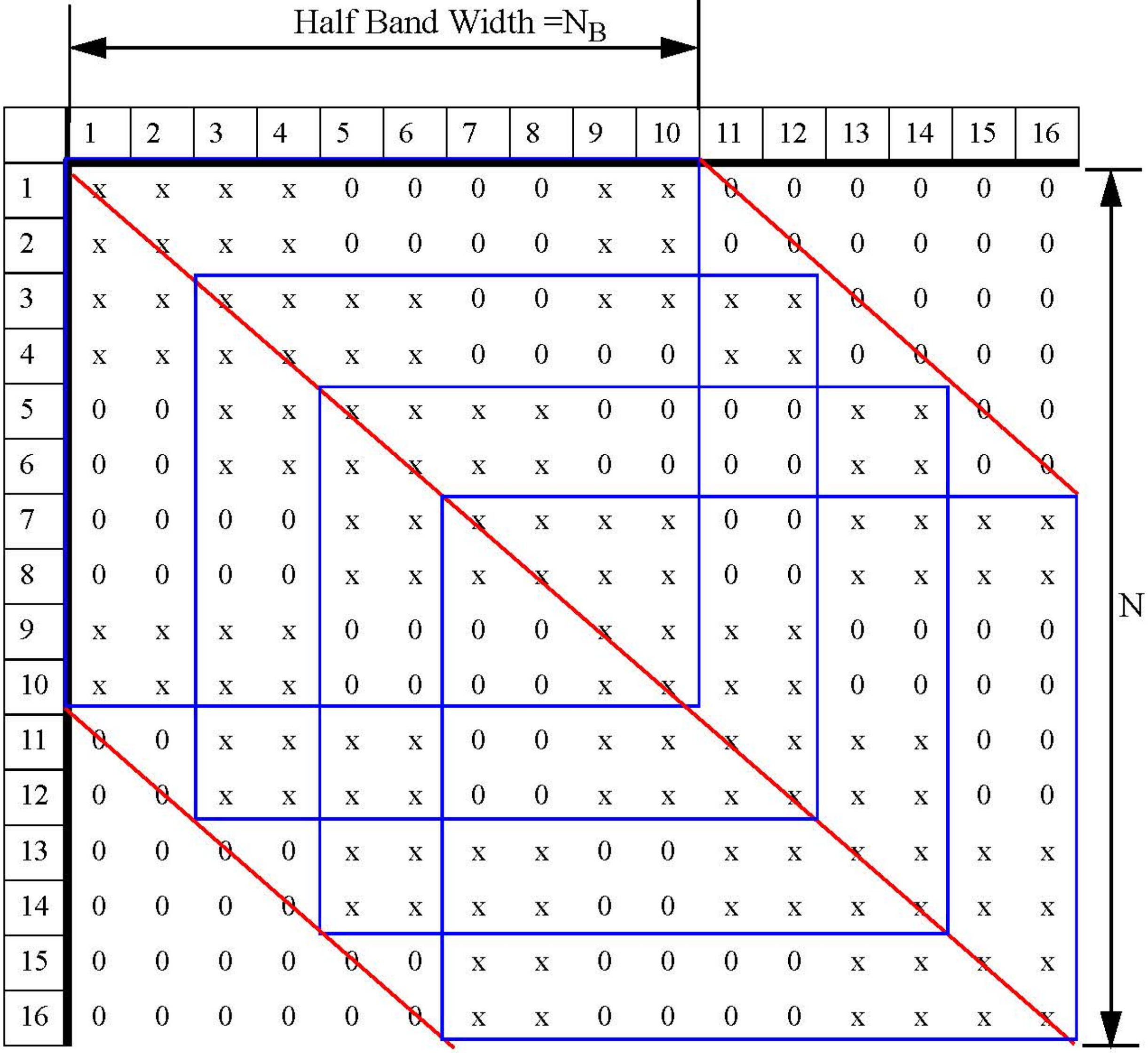
2	x	x	x	x	x	x	0	0	0	0	0	0	0	0	0	0
3	x	x	x	x	x	x	x	x	0	0	0	0	0	0	0	0
4	x	x	x	x	x	x	x	x	0	0	0	0	0	0	0	0
5	x	x	x	x	x	x	x	x	x	x	0	0	0	0	0	0
6	x	x	x	x	x	x	x	x	x	x	0	0	0	0	0	0
7	0	0	x	x	x	x	x	x	x	x	x	x	0	0	0	0
8	0	0	x	x	x	x	x	x	x	x	x	x	0	0	0	0
9	0	0	0	0	x	x	x	x	x	x	x	x	x	x	0	0
10	0	0	0	0	x	x	x	x	x	x	x	x	x	x	0	0
11	0	0	0	0	0	0	x	x	x	x	x	x	x	x	x	x
12	0	0	0	0	0	0	x	x	x	x	x	x	x	x	x	x
13	0	0	0	0	0	0	0		x	x	x	x	x	x	x	x
14	0	0	0	0	0	0	0	0	x	x	x	x	x	x	x	x
15	0	0	0	0	0	0	0	0	0	0	x	x	x	x	x	x
16	0	0	0	0	0	0	0	0	0	0	x	x	x	x	x	x

- Only the upper banded form of the global stiffness matrix is stored in a rectangular matrix of size NxN_B

Banded Matrix





	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	x	x	x	x	0	0	0	0	x	x	0	0	0	0	0	0
2	x	x	x	x	0	0	0	0	x	x	0	0	0	0	0	0
3	x	x	x	x	x	x	0	0	x	x	x	x	0	0	0	0
4	x	x	x	x	x	x	0	0	0	0	x	x	0	0	0	0
5	0	0	x	x	x	x	x	x	0	0	0	0	x	x	0	0
6	0	0	x	x	x	x	x	x	0	0	0	0	x	x	0	0
7	0	0	0	0	x	x	x	x	x	x	0	0	x	x	x	x
8	0	0	0	0	x	x	x	x	x	x	0	0	x	x	x	x
9	x	x	x	x	0	0	0	0	x	x	x	x	0	0	0	0
10	x	x	x	x	0	0	0	0	x	x	x	x	0	0	0	0
11	0	0	x	x	x	x	0	0	x	x	x	x	x	x	0	0
12	0	0	x	x	x	x	0	0	x	x	x	x	x	x	0	0
13	0	0	0	0	x	x	x	x	0	0	x	x	x	x	x	x
14	0	0	0	0	x	x	x	x	0	0	x	x	x	x	x	x
15	0	0	0	0	0	0	x	x	0	0	0	0	x	x	x	x
16	0	0	0	0	0	0	x	x	0	0	0	0	x	x	x	x

- The bandwidth is solely dictated by the difference between the highest and the lowest node number on the elements.

Equation Solving [K]{d}={R}

Direct Methods: Matrix inversion, Gauss Elimination, Gauss-Jordan, Cholesky's Method.
Iterative Methods: Jacobi, Gauss-Seidel, Successive Overrelaxation.

Gauss Elimination

- A lower triangular matrix has zero coefficients above the diagonal [L]
- A upper triangular matrix has zero coefficients below the diagonal [U]

Any matrix [K] can be decomposed as [K]=[L][U]. The algebraic equations become: [L][U]{d}={R}. If we substitute [U]{d}={x} we obtain [L]{x}={R}. The solution is obtained in two steps:

1. Forward substitution: Solve for {x} from: [L]{x}={R}
2. Backward substitution: Solve for {d} from: [U]{d}={x}

- In FEM the stiffness matrix is symmetric and positive definite. The decomposition is done using Cholesky's method to obtain $[U]=[L]^T$.
- Forward substitution is done in increasing node number order and backward substitution is done in decreasing node number order.

Pros and Cons

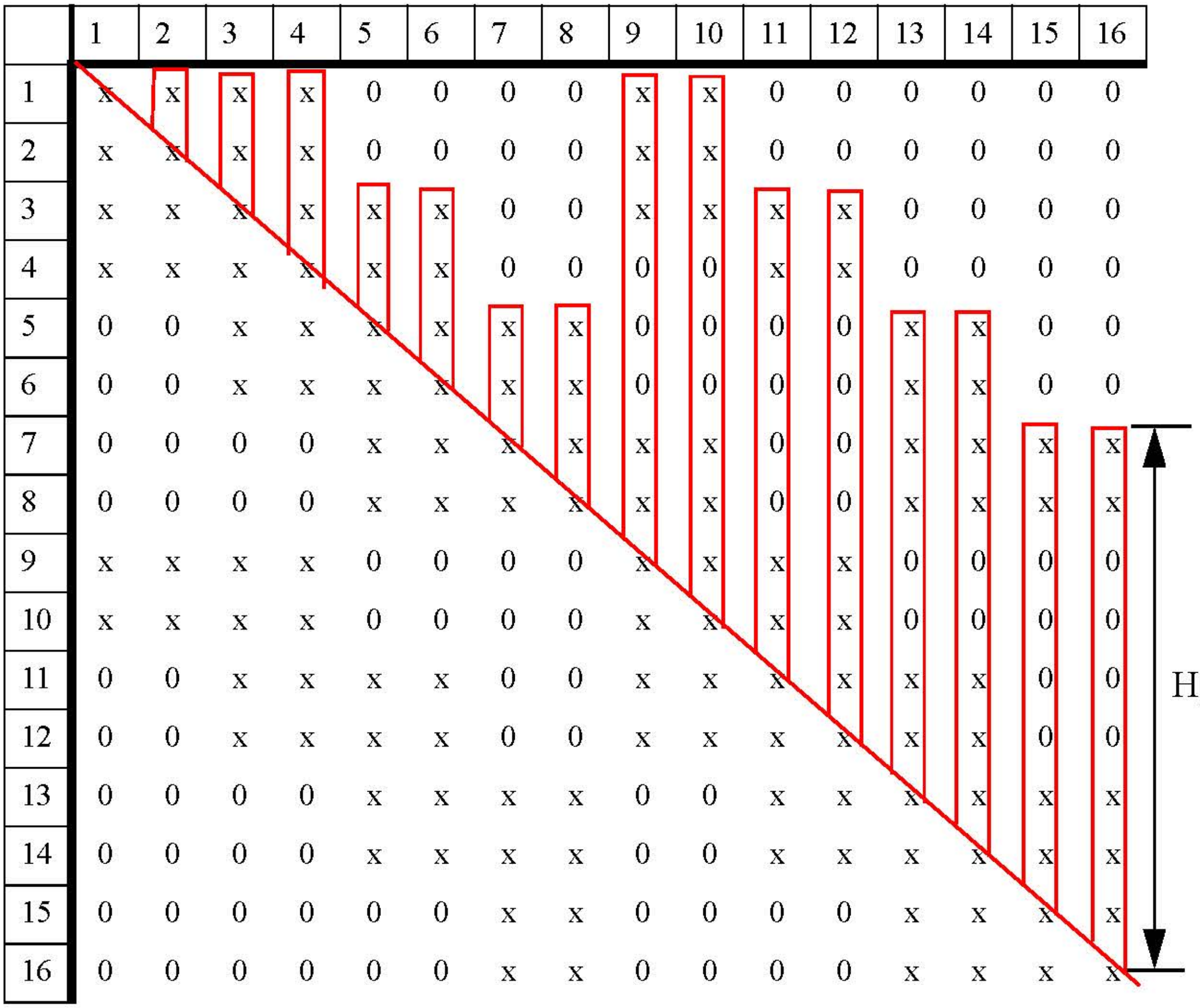
- Direct methods are better for multiple load vectors.The computation time is proportional to NN_B^2.
- Computation times for iterative methods depend upon the starting guess solution. In time dependent problems that are solved in small time steps the initial guess vector is known from previous time step. Non-linear problems which are solved as linear problems iteratively also use less time in iterative methods.

Large Systems

Skyline storage scheme

The stiffness matrix of very large structures may have mostly zero terms, even inside the band width. Such matrices are called sparse matrices.

H_i= The height of the column above the diagonal of the ith row beyond which there is no non-zero term.



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	x	x	x	x	0	0	0	0	x	x	0	0	0	0	0	0
2	x	x	x	x	0	0	0	0	x	x	0	0	0	0	0	0
3	x	x	x	x	x	x	0	0	x	x	x	x	0	0	0	0
4	x	x	x	x	x	x	0	0	0	0	x	x	0	0	0	0
5	0	0	x	x	x	x	x	x	0	0	0	0	x	x	0	0
6	0	0	x	x	x	x	x	x	0	0	0	0	x	x	0	0
7	0	0	0	0	x	x	x	x	x	x	0	0	x	x	x	x
8	0	0	0	0	x	x	x	x	x	x	0	0	x	x	x	x
9	x	x	x	x	0	0	0	0	x	x	x	x	0	0	0	0
10	x	x	x	x	0	0	0	0	x	x	x	x	0	0	0	0
11	0	0	x	x	x	x	0	0	x	x	x	x	x	x	0	0
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13	0	0	0	0	x	x	x	x	0	0	x	x	x	x	x	x
14	0	0	0	0	x	x	x	x	0	0	x	x	x	x	x	x
15	0	0	0	0	0	0	x	x	0	0	0	0	x	x	x	x
16	0	0	0	0	0	0	x	x	0	0	0	0	x	x	x	x

Skyline algorithms are used for sparse matrices.

Wavefront (Frontal) equation solver:

- Assembly and equation solving proceeds simultaneously.
- Computation of coefficients in a stiffness matrix and load vector is complete only when contribution of all elements that share the node have been added.
- A complete coefficient is can be processed in the solution process.
- Wavefront number is a measure of number of coefficients being manipulated in the solution process at a given step.

Substructure (Matrix partitioning)

Algebraic equations can be written as: $$\left[\begin{array}{c|c} [A] & [B] \\ \hline [C] & [D] \end{array}\right] \left\{\begin{array}{c} \{d_1\} \\ \hline \{d_2\} \end{array}\right\} = \left\{\begin{array}{c} \{R_1\} \\ \hline \{R_2\} \end{array}\right\}$$

The two sets of matrix equations can be solved as shown below.

$$[A]\{d_1\} + [B]\{d_2\} = \{R_1\} \quad \text{or} \quad \{d_1\} = [A]^{-1}(\{R_1\} - [B]\{d_2\})$$

$$[C]\{d_1\} + [D]\{d_2\} = \{R_2\}$$

$$[C][A]^{-1}(\{R_1\} - [B]\{d_2\}) + [D]\{d_2\} = \{R_2\}$$

$$([D] - [C][A]^{-1}[B])\{d_2\} = \{R_2\} - [C][A]^{-1}\{R_1\}$$

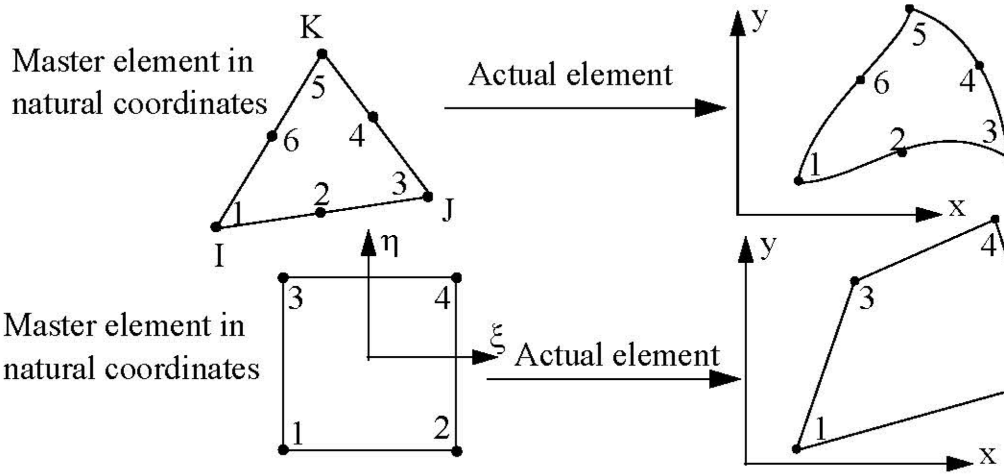
$$[K_S]\{d_2\} = \{R_S\}$$

In substructuring $\{d_2\}$ are the nodal displacements on the interface of two substructures and $\{d_1\}$are the displacements of nodes on either of the substructure.

- Substructuring is used for large structures. Each substructure can be used as a superelement if a substructure repeats in a structure.
- Substructuring could be used for creating meshes with different densities either for creating original mesh or for mesh refinements.
- In non-linear analysis such as in elastic-plastic analysis. The plastic zone is made a part of a substructure.

Isoparametric Elements

- Coordinates of the geometry is approximated using the same interpolation functions as used for displacements.



Master element in natural coordinates

Actual element

Master element in natural coordinates

Actual element

$$u = \sum_{i=1}^{n} N_i(\xi,\eta)u_i^{(e)} \qquad v = \sum_{i=1}^{n} N_i(\xi,\eta)v_i^{(e)} \qquad x = \sum_{i=1}^{m} N_i(\xi,\eta)x_i \qquad y = \sum_{i=1}^{m} N_i(\xi,\eta)y_i$$

1. Subparametric elements $m < n$
2. Isoparametric elements $m = n$---Most common
3. Superparametric elements $m > n$---Rarely if ever used.

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \sum_{i=1}^{m} \dfrac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{m} \dfrac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{m} \dfrac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{m} \dfrac{\partial N_i}{\partial \eta} y_i \end{bmatrix}}_{[J]} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} \qquad \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix}$$

$$dx = \left(\sum_{i=1}^{---} \frac{\partial N_i}{\partial \xi} x_i\right) d\xi + \left(\sum_{i=1}^{---} \frac{\partial N_i}{\partial \eta} x_i\right) d\eta$$

$$dy = \left(\sum^{m} \frac{\partial N_i}{\partial \xi} y_i\right) d\xi + \left(\sum^{m} \frac{\partial N_i}{\partial \eta} y_i\right) d\eta$$

$$\begin{Bmatrix} dx \\ dy \end{Bmatrix} = [J]^T \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} \qquad dxdy = |J| d\xi d\eta$$

$$[K^{(e)}] = \iint [B]^T [E][B] t (dx)(dy) = \int_{-1}^{1}\int_{-1}^{1} [\tilde{B}]^T [E][\tilde{B}] t |J| (d\xi)(d\eta)$$

Numerical Integration

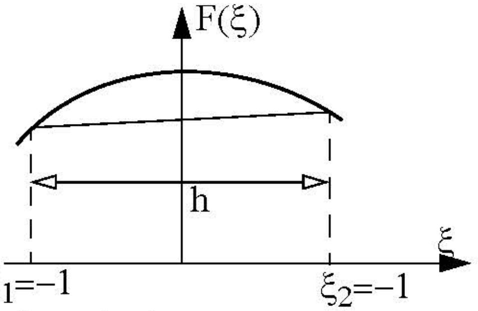
$$I = \int_{-1}^{1} F(\xi)d\xi = \sum_{i=1}^{n} w_i F(\xi_i)$$

- w_i are called the weights and ξ_i are called the base points. Functions are evaluated at base points.

Trapezoidal rule: (n=2, linear approximation of F(ξ)

$$I = \int_{-1}^{1} F(\xi)d\xi = \frac{h}{2}(F(\xi_1) + F(\xi_2))$$

$$w_1 = w_2 = \frac{h}{2}$$



Simpson's rule (n=3, quadratic approximation of F(ξ)):

$$I = \int_{-1}^{1} F(\xi)d\xi = \frac{h}{3}(F(\xi_1) + 4F(\xi_2) + F(\xi_3))$$

Newton-Cotes quadrature: Equally spaced base points. For $(n\text{-}1)^{th}$ order polynomial is integrated exactly by choosing n base points. Note there are n parameters (weights) that can be adjusted.

Gauss quadrature (Gauss-Legendre quadrature)

Make base points also variables, thus there are 2n parameters, hence a polynomial of order (2n-1) will be integrated exactly. Using this idea the weights and base points are determined. It turns out that the base points are the roots of Legendre Polynomials.

Table 1: Weights and Gauss (base) points

Base points ξ_i	n	Weights w_i
0.0	One point formula	$w_1 = 2.0$
$\pm(1/\sqrt{3})$	Two point formula	$w_1 = w_2 = 1.0$
0.0 $\pm\sqrt{0.6}$	Three point formula	$w_1 = (8/9)$ $w_2 = w_3 = (5/9)$

2-D Numerical interrogation

$$I = \int_{-1}^{1}\int_{-1}^{1} F(\xi, \eta)(d\xi)d\eta = \sum_{j=1}^{n}\sum_{i=1}^{n} w_i w_j F(\xi_i, \eta_j)$$

Class Problem

Evaluate the integral below using 1,2, and 3 points Gauss Quadrature

$$I = \int_{-1}^{1}\int_{-1}^{1} \frac{(\xi^2 + \xi + 2)}{1 + \eta^2} d\xi d\eta$$

Homework Problem

Evaluate the integral below using 1,2, and 3 points Gauss Quadrature

$$I = \int_{-1}^{1}\int_{-1}^{1} \frac{(1 + \xi^2)}{2 + \eta^2} d\xi d\eta$$

Class Problem: Evaluate the integral below using 1,2, and 3 points Gauss Quadrature $I = \int_{-1}^{1}\int_{-1}^{1}\frac{(\xi^2+\xi+2)}{1+\eta^2}d\xi d\eta$

Analytical value: $I = \left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+2\xi\right)\operatorname{atan}(\eta)\Big|_{-1}^{1}\Big|_{-1}^{1} = 7.3303$

Table 2: Class Problem Solution

ξ_i	η_j	$F(\xi_i, \eta_j)$	w_i	w_j	$w_i w_j F(\xi_i, \eta_j)$
One point Gauss Quadrature					
0	0	2	2	2	8
Two point Gauss Quadrature					
-0.5774	-0.5774	1.3170	1.0000	1.0000	1.3170
0.5774	-0.5774	2.1830	1.0000	1.0000	2.1830
-0.5774	0.5774	1.3170	1.0000	1.0000	1.3170
0.5774	0.5774	2.1830	1.0000	1.0000	2.1830
				Total I=	7.0000
Three point Gauss Quadrature					
-0.7746	-0.7746	1.1409	0.5556	0.5556	0.3521
0.0000	-0.7746	1.2500	0.8889	0.5556	0.6173
0.7746	-0.7746	2.1091	0.5556	0.5556	0.6510
-0.7746	0.0000	1.8254	0.5556	0.8889	0.9014
0.0000	0.0000	2.0000	0.8889	0.8889	1.5802
0.7746	0.0000	3.3746	0.5556	0.8889	1.6665
-0.7746	0.7746	1.1409	0.5556	0.5556	0.3521
0.0000	0.7746	1.2500	0.8889	0.5556	0.6173
0.7746	0.7746	2.1091	0.5556	0.5556	0.6510
				Total I=	7.3889

- More Gauss points increases the accuracy of integration but may make the element more stiff thus decreasing the FEM accuracy.
- Two few Gauss points may cause instabilities resulting in singular stiffness matrices.
- Evaluation of stresses at Gauss points is usually more accurate than at nodes.

FEM Convergence

1. Does the FEM solution converge as mesh is refined?
2. Does the FEM solution converge rapidly?
3. Does the FEM solution converge to the right solution?

- Minimum potential energy theorem guarantees convergence provided the set of kinematically admissible functions are complete and independent.
- Internal nodes in an element are needed to ensure the set of interpolation functions are complete.
- Internal nodes in an element do not play as significant a role as nodes on the boundaries of an element and very often are eliminated to accelerate convergence. Need a test to ensure the FEM solution will still converge.

Patch Test

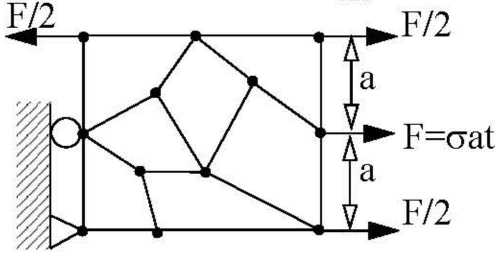
A numerical test designed to check if a mesh made from a specific type of element will converge as it is refined.

1. Create a patch (mesh) from the element that is to be tested. Must have at least one node in the interior of the patch.
2. Apply minimum boundary conditions to eliminate rigid body mode.
3. Apply loading to produce constant stress inside the patch.

The test is successful if it displays:

1. A constant strain inside the patch.
2. Deformation on the element boundary of element satisfies compatibility i.e., continuity requirements—no holes for plane problems and no holes and corners for plate problems.
3. Rigid body motion without strain.

Constant stress=$\sigma_{xx} = \sigma$



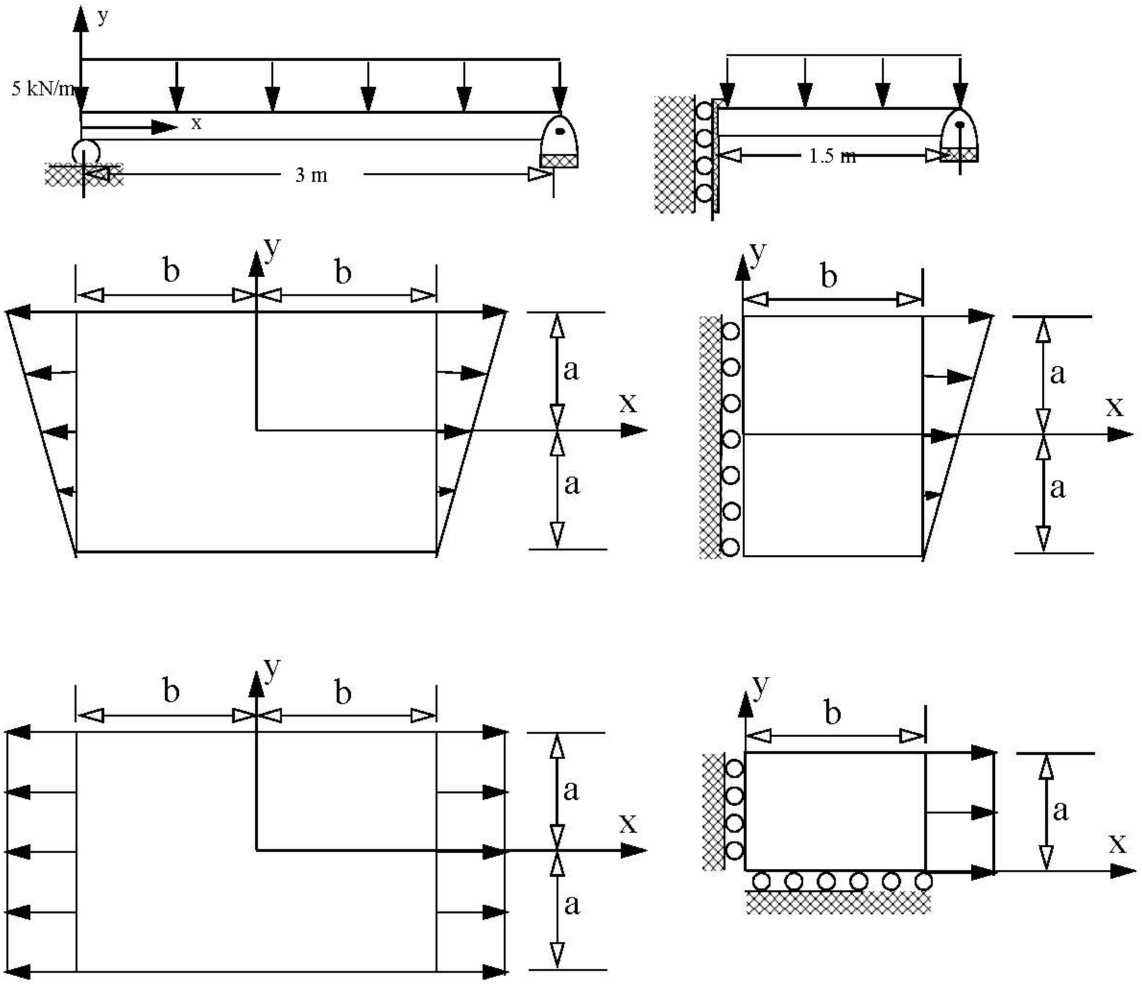
- It is user's responsibility to ensure the solution converges to the right solution at a satisfactory convergence rate.

Symmetry

- Symmetry can be used to reduce the model size in the analysis.

Symmetry about a plane (axis) requires:

1. Symmetry of geometry.
2. Symmetry in loading
3. Symmetry of material properties.


y
5 kN/m
x
3 m
1.5 m
b
y
b
a
x
a
y
b
a
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b
y
b
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y
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