Introduction to Finite Element Method

There are two versions of FEM

1. Flexibility Method or Force Method:
2. Stiffness Method or Displacement Method.

- The set of equations in the stiffness method are the equilibrium equations relating displacements of points.
- Rayleigh-Ritz is an approximate method based on energy principle by which we can obtain equilibrium equations in matrix form.

The learning objectives in this chapter is:

1. Understand Rayleigh-Ritz method and its application to axial members, torsion of circular shafts, and symmetric bending of beams.
2. Understand the perspective, the key issues, the terminology, and steps used in solving problems by finite element method.

Rayleigh Ritz

- Rayleigh Ritz method is an approximate method of finding displacements that is based on the theorem of minimum potential energy.
- The method is restricted to conservative systems that may be linear or non-linear.

Potential energy function (Ω): $\Omega = U - W$

U is the strain energy
W is the work potential of a force.

Internal virtual work: $\delta W_{int} = \delta U$

External virtual work: $\delta W_{ext} = \delta W$

Virtual work principle: $\delta W_{int} - \delta W_{ext} = \delta U - \delta W = \delta\Omega = 0$

- The zero virtual variation in the potential energy function occurs where the slope of the potential energy function with respect to the parameters defining the potential function is zero.
- The parameters defining the potential function are the generalized displacements.

The zero slope condition can occur where the potential function is:

maximumUnstable equilibrium

saddle pointNeutral equilibrium

minimumStable equilibrium

Minimum Potential Energy

The theorem of minimum potential energy can be stated as:

Of all the kinematically admissible displacement functions the actual displacement function is the one that minimizes the potential energy function at stable equilibrium.

Corollary

- The better approximation of displacement function is the one that yields a lower potential energy.
- The greater the degrees of freedom, lower will be the potential energy for a given set of kinematically admissible functions.

Procedure

Let the displacement u(x) be represented by a series of kinematically admissible functions $f_i(x)$, i.e.,

$$u(x) = \sum_{i=1}^{n} C_i f_i(x)$$

Substitute in Potential Energy: $\Omega = \Omega(C_1, C_2, C_3 \ldots\ldots C_n)$

Minimize Potential energy:

$$\frac{\partial \Omega}{\partial C_i} = 0 \qquad i = 1, n$$

Set of n-algebraic equations in the unknown C_i.

Rayleigh Ritz for Axial Members

Axial strain energy density:

$$U_a = \frac{1}{2}EA\left(\frac{du}{dx}\right)^2$$

Displacement Approximation: $\boxed{u(x) = \sum_{i=1}^{n} C_i f_i(x)}$

$$\frac{du}{dx} = \sum_{j=1}^{n} C_j \frac{df_j}{dx}$$

$$U_a = \frac{1}{2}EA\left(\frac{du}{dx}\right)\left(\frac{du}{dx}\right) = \frac{1}{2}EA\left(\sum_{j=1}^{n} C_j \frac{df_j}{dx}\right)\left(\sum_{k=1}^{n} C_k \frac{df_k}{dx}\right)$$

$$U_a = \frac{1}{2}EA\sum_{j=1}^{n}\sum_{k=1}^{n} C_j C_k \left(\frac{df_j}{dx}\right)\left(\frac{df_k}{dx}\right)$$

Axial strain energy

$$U_A = \int_0^L U_a dx = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} C_j C_k \left[\int_0^L EA\left(\frac{df_j}{dx}\right)\left(\frac{df_k}{dx}\right)dx\right]$$

Define: $\boxed{K_{jk} = \int_0^L EA\left(\frac{df_j}{dx}\right)\left(\frac{df_k}{dx}\right)dx}$ ----Stiffness matrix

- EA can be function of x.
- $K_{jk} = K_{kj}$ Stiffness Matrix is symmetric

Axial Strain Energy: $$U_A = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} C_j C_k K_{jk}$$

Work potential

$p_x(x)$ axial distributed force

F_q concentrated axial forces applied at m x_q points

The work potential of these forces can be written as:

$$W_A = \int_0^L p_x(x)u(x)dx + \sum_{q=1}^{m} F_q u(x_q)$$

Substituting the displacement approximation.

$$W_A = \sum_{j=1}^{n} C_j \int_0^L p_x(x) f_j(x)\ dx + \sum_{q=1}^{m}\sum_{j=1}^{n} F_q C_j f_j(x_q)$$

Define: $$\boxed{R_j = \int_0^L p_x(x) f_j(x)\ dx + \sum_{q=1}^{m} F_q f_j(x_q)}$$ ---Load Vector

$$W_A = \sum_{j=1}^{n} C_j R_j$$

Axial potential energy function:

$$\Omega_A = U_A - W_A = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} C_j C_k K_{jk} - \sum_{j=1}^{n} C_j R_j$$

Axial potential energy function:

$$\Omega_A = U_A - W_A = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} C_j C_k K_{jk} - \sum_{j=1}^{n} C_j R_j$$

Minimize the potential energy function: $\dfrac{\partial \Omega_A}{\partial C_i} = 0$

$$\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\frac{\partial C_j}{\partial C_i}C_k + C_j\frac{\partial C_k}{\partial C_i}\right]K_{jk} - \sum_{j=1}^{n}\frac{\partial C_j}{\partial C_i}R_j = 0 \qquad i = 1, n$$

Note: $\dfrac{\partial C_j}{\partial C_i} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$$\frac{1}{2}\left(\sum_{k=1}^{n} C_k K_{ik} + \sum_{j=1}^{n} C_j K_{ji}\right) - R_i = 0 \qquad i = 1, n$$

$$\frac{1}{2}\left(\sum_{j=1}^{n} C_j (K_{ij} + K_{ji})\right) - R_i = 0 \qquad i = 1, n$$

Using symmetry of stiffness matrix: $K_{ji} = K_{ij}$

$$\boxed{\sum_{j=1}^{n} K_{ij} C_j = R_i \qquad i = 1, n} \quad \text{or} \quad \boxed{[K]\{C\} = \{R\}}$$

At Equilibrium only

$$\Omega_A = \frac{1}{2}\sum_{j=1}^{n} C_j\left(\sum_{k=1}^{n} K_{jk} C_k\right) - \sum_{j=1}^{n} C_j R_j$$

$$\Omega_A = \frac{1}{2}\sum_{j=1}^{n} C_j R_j - \sum_{j=1}^{n} C_j R_j = -\left(\frac{1}{2}\right)\sum_{j=1}^{n} C_j R_j$$

Strain energy at equilibrium

$$U_A = \frac{W_A}{2} = -(\Omega_A) = \frac{1}{2}\sum_{j=1}^{n} C_j R_j \qquad \text{at equilibrium}$$

- More the degree of freedom, lower is the potential energy (more negative), higher is the strain energy stored in the structure.
- With greater degree of freedom the structure becomes less stiff and hence can store more energy.
- In FEM mesh refinement schemes try to get a more uniform distribution of strain energy across the structure.

Symmetric Bending of Beams

Bending strain energy density: $U_b = \frac{1}{2} EI_{zz} \left(\frac{d^2 v}{dx^2} \right)^2$

Displacement Approximation: $v(x) = \sum_{i=1}^{n} C_i f_i(x)$

Work Potential:

$p_y(x)$ distributed load

F_q concentrated transverse forces applied at m_1 x_q points

M_q concentrated transverse forces applied at m_2 x_q points

$$W_A = \int_0^L p_y(x)\phi(x)dx + \sum_{q=1}^{m_1} F_q v(x_q) + \sum_{q=1}^{m_2} M_q \frac{dv}{dx}(x_q)$$

Stiffness Matrix:

$$\boxed{K_{jk} = \int_0^L (EI_{zz}) \left(\frac{d^2 f_j}{dx^2} \right) \left(\frac{d^2 f_k}{dx^2} \right) dx}$$

Load Vector:

$$\boxed{R_j = \int_0^L p_y(x) f_j(x)\ dx + \sum_{q=1}^{m_1} F_q f_j(x_q) + \sum_{q=1}^{m_1} M_q \frac{df_j}{dx}(x_q)}$$

8.6 For the beam shown in Fig. P8.6 determine one and two parameter Rayleigh-Ritz displacement solution using the approximation for the bending displacement given below. Calculate the potential energy function for the one and two parameters.

$$v(x) = \sum_{i=1}^{n} C_i \sin\frac{i\pi x}{2L}$$

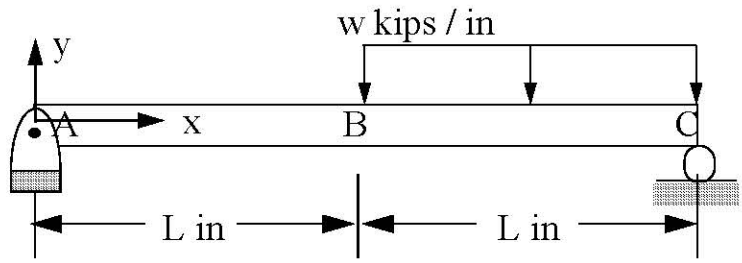


Fig. P8.6

Finite Element Method (FEM)

In Rayleigh-Ritz the kinematically admissible function is over the entire structure. In FEM the kinematically admissible function is piecewise continuous over small (finite) domains called the 'elements'.
Potential energy is a scaler quantity and can be written as:

$$\delta\Omega = \sum_{i=1}^{n} \delta\Omega^{(i)}$$

where, n is the number of elements (structural members).

- This perspective of reducing the *complexity of analysis of large structures to analysis of simple individual members (elements)* is what makes the finite element method such a versatile and popular analysis tool in structural (and engineering application) analysis.

Definition 1 Nodes are points on the structure at which displacements and rotations are to be found or prescribed.

Definition 2 Element is a small domain on which we can solve the boundary value problem in terms of the displacements and forces of the nodes on the element.

Definition 3 The discrete representation of the structure geometry by elements and nodes is called a mesh.

Definition 4 The process of creating a mesh (discrete entities) is called discretization.

Definition 5 Interpolation function is a kinematically admissible displacement function defined on an element that can be used for interpolating displacement values between the nodes.

Definition 6 The mesh, boundary conditions, loads, and material properties representing the actual structure is called a model.

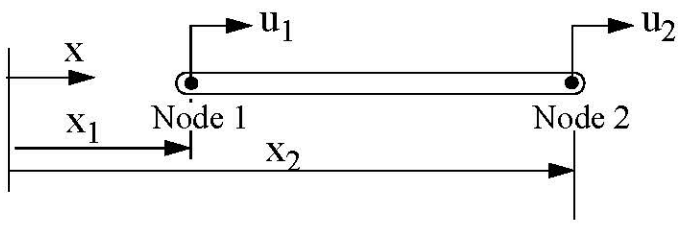
Definition 7 Element stiffness matrix relate the displacements to the forces at the element nodes.

Definition 8 Global stiffness matrix is an assembly of element stiffness matrix that relates the displacements of the nodes on the mesh to applied external forces.

Lagrange polynomials

- Polynomial functions that yield function value at finite number of points.

Linear Approximation: $u(x) = C_1 + C_2 x$

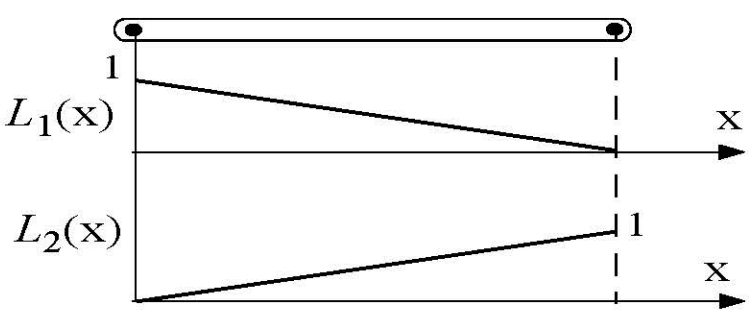


Conditions: $u(x_1) = u_1 \qquad u(x_2) = u_2$

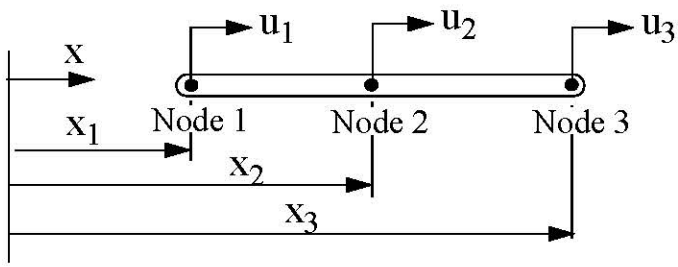
$$u(x) = u_1\left(\frac{x - x_2}{x_1 - x_2}\right) + u_2\left(\frac{x - x_1}{x_2 - x_1}\right)$$

Define: $\quad L_1(x) = \left(\frac{x - x_2}{x_1 - x_2}\right) \qquad L_2(x) = \left(\frac{x - x_1}{x_2 - x_1}\right)$

$$u(x) = u_1 L_1(x) + u_2 L_2(x) = \sum_{i=1}^{2} u_i L_i(x)$$



Quadratic displacement function: $u(x) = C_1 + C_2 x + C_3 x^2$



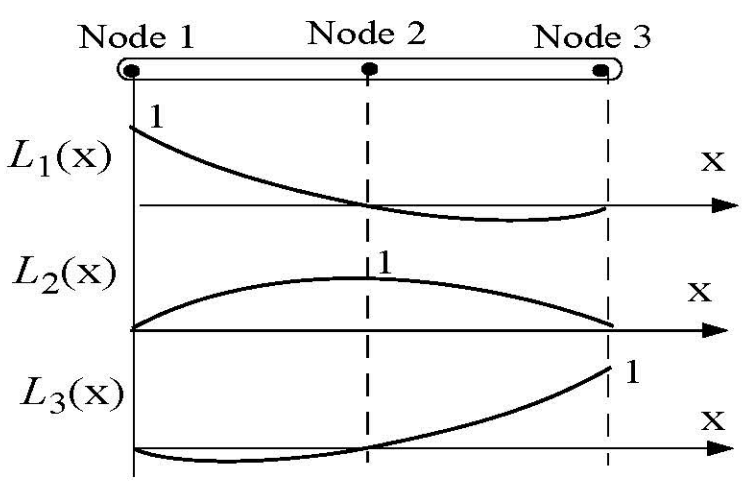
Alternative Approach: $u(x) = \sum_{i=1}^{n} u_i L_i(x)$

$$u(x_j) = \sum_{i=1}^{n} u_i L_i(x_j) = u_j \qquad L_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

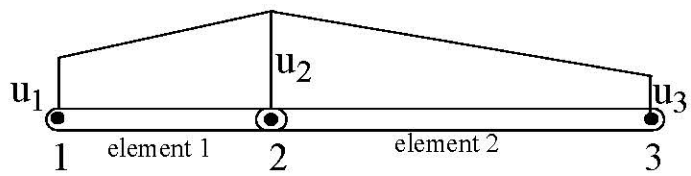
$$L_1(x) = a_1(x - x_2)(x - x_3) \qquad L_1(x) = \left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right)$$

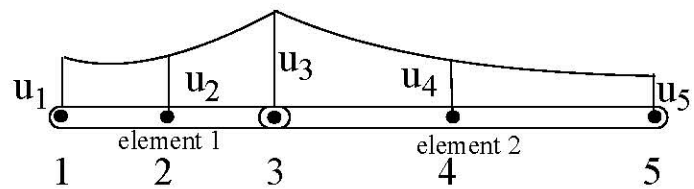
$$L_2(x) = a_2(x - x_3)(x - x_1) \qquad L_2(x) = \left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_3}{x_2 - x_3}\right)$$

$$L_3(x) = a_3(x - x_1)(x - x_2) \qquad L_3(x) = \left(\frac{x - x_1}{x_3 - x_1}\right)\left(\frac{x - x_2}{x_3 - x_2}\right)$$



nth order polynomial: $$L_i(x) = \prod_{\substack{j=1 \\ i \neq j}}^{n} \left[\frac{(x - x_j)}{(x_i - x_j)} \right] \qquad i = 1, (n+1)$$





- Lagrange polynomials ensure that the function these are used to represent will be continuous at the element ends.
- The continuity of the derivative of the function cannot be ensured at the element end irrespective of the order of polynomials when Lagrange polynomials are used for representing the function.

Axial elements

Notation: superscript to designate quantities at element level and no superscript at the global level. $u_3^{(1)}$ and $u_3^{(2)}$ l refer to displacement of node 3 for element 1 and 2. u_3 refers to displacement of node 3 on the actual structure.

Displacements:

$$u^{(e)} = \sum_{i=1}^{n} u_i^{(e)} L_i(x)$$

Axial strains:

$$\varepsilon_{xx}^{(e)} = \frac{du^{(e)}}{dx} = \sum_{i=1}^{n} u_i^{(e)} \frac{dL_i}{dx}$$

Axial Stress:

$$\sigma_{xx}^{(e)} = E^{(e)} \varepsilon_{xx}^{(e)} = E^{(e)} \sum_{i=1}^{n} u_i^{(e)} \frac{dL_i}{dx}$$

Internal axial force:

$$N^{(e)} = A^{(e)} \sigma_{xx}^{(e)} = E^{(e)} A^{(e)} \sum_{i=1}^{n} u_i^{(e)} \frac{dL_i}{dx}$$

In Equations for Rayleigh-Ritz substitute: $c_i = u_i^{(e)}$ and $f_i(x) = L_i(x)$

Element stiffness matrix:

$$K_{jk}^{(e)} = \int_0^L E^{(e)} A^{(e)} \left(\frac{dL_j}{dx}\right)\left(\frac{dL_k}{dx}\right) dx$$

Element load vector:

$$R_j^{(e)} = \int_0^L p_x(x) L_j(x) \; dx + F_1^{(e)} L_j(x_1) + F_n^{(e)} L_j(x_n)$$

For $p_x = 0$ $\quad R_1^{(e)} = F_1^{(e)} \quad R_n^{(e)} = F_n^{(e)} \quad R_j^{(e)} = 0 \quad j = 2, (n-1)$

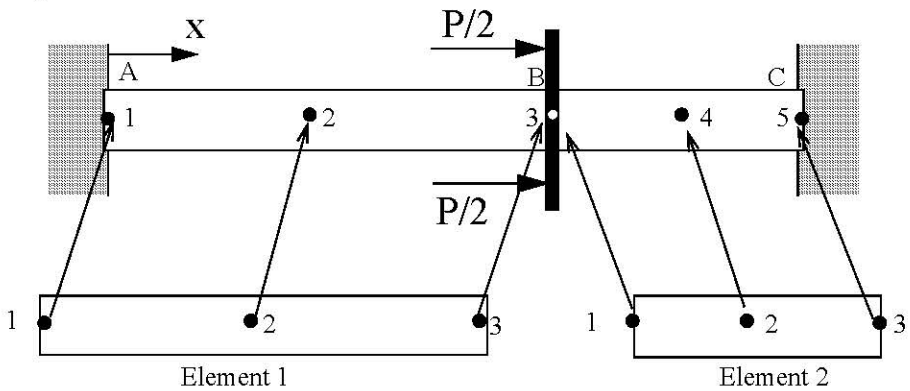
- $F_j^{(e)}$ is positive in the positive direction $u_j^{(e)}$.

Assembly of global matrix and global load vector

- The governing criteria is that the displacement function at the node where two elements meet must be continuous i.e., be the same.

Assume there is no distributed force i.e., $p_x = 0$

Use two quadratic axial elements



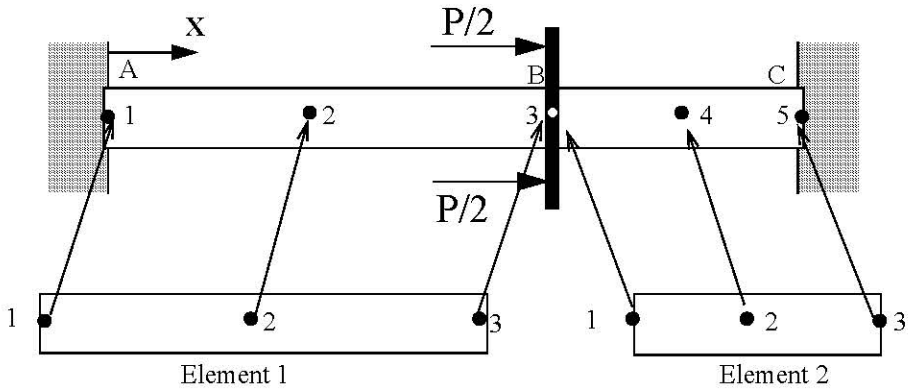
Virtual variation in potential energy in an element:

$$\delta\Omega^{(m)} = \sum_{i=1}^{n} \frac{\partial\Omega^{(m)}}{\partial u_i^{(m)}} \delta u_i^{(m)}$$

$$\delta\Omega^{(m)} = \sum_{i=1}^{n} \delta u_i^{(m)} \sum_{j=1}^{n} (K_{ij}^{(m)} u_j^{(m)} - R_i^{(m)})$$

$$\delta\Omega^{(1)} = \begin{Bmatrix} \delta u_1^{(1)} \\ \delta u_2^{(1)} \\ \delta u_3^{(1)} \end{Bmatrix}^T \left(\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{Bmatrix} - \begin{Bmatrix} F_1^{(1)} \\ 0 \\ F_3^{(1)} \end{Bmatrix} \right)$$

$$\delta\Omega^{(2)} = \begin{Bmatrix} \delta u_1^{(2)} \\ \delta u_2^{(2)} \\ \delta u_3^{(2)} \end{Bmatrix}^T \left(\begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} \begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_3^{(2)} \end{Bmatrix} - \begin{Bmatrix} F_1^{(2)} \\ 0 \\ F_3^{(2)} \end{Bmatrix} \right)$$



$$u_1^{(1)} = u_1 \qquad u_2^{(1)} = u_2 \qquad u_3^{(1)} = u_3$$

$$u_1^{(2)} = u_3 \qquad u_2^{(2)} = u_4 \qquad u_3^{(2)} = u_5$$

$$\delta\Omega^{(1)} = \begin{Bmatrix} \delta u_1^{(1)} \\ \delta u_2^{(1)} \\ \delta u_3^{(1)} \end{Bmatrix}^T \left(\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{Bmatrix} - \begin{Bmatrix} F_1^{(1)} \\ 0 \\ F_3^{(1)} \end{Bmatrix} \right)$$

$$\delta\Omega^{(1)} = \begin{Bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \\ \delta u_4 \\ \delta u_5 \end{Bmatrix}^T \left(\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & 0 & 0 \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} - \begin{Bmatrix} F_1^{(1)} \\ 0 \\ F_3^{(1)} \\ 0 \\ 0 \end{Bmatrix} \right)$$

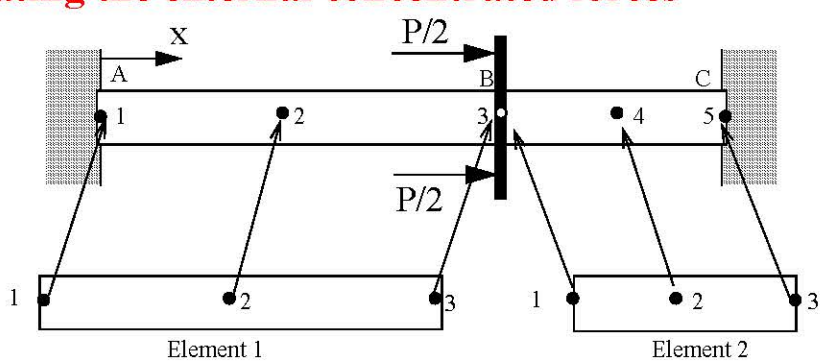
$$\delta\Omega^{(2)} = \begin{Bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \\ \delta u_4 \\ \delta u_5 \end{Bmatrix}^T \left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} \\ 0 & 0 & K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ 0 & 0 & K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ F_1^{(2)} \\ 0 \\ F_3^{(2)} \end{Bmatrix} \right)$$

Total potential energy $\delta\Omega = \delta\Omega^{(1)} + \delta\Omega^{(2)}$

$$\delta\Omega = \begin{Bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \\ \delta u_4 \\ \delta u_5 \end{Bmatrix}^T \left(\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & 0 & 0 \\ K_{31}^{(1)} & K_{32}^{(1)} & (K_{33}^{(1)} + K_{11}^{(2)}) & K_{12}^{(2)} & K_{13}^{(2)} \\ 0 & 0 & K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ 0 & 0 & K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} - \begin{Bmatrix} F_1^{(1)} \\ 0 \\ F_3^{(1)} + F_1^{(2)} \\ 0 \\ F_3^{(2)} \end{Bmatrix} \right)$$

- The element stiffness matrix and element load components that add corresponds to the degree of freedom associated with the shared node of the elements.

Incorporating the external concentrated forces



The force $F_3^{(1)}$ is in the direction of $u_3^{(1)}$

The force $F_1^{(2)}$ is in the direction of $u_1^{(2)}$

The applied force is in the direction of u_3.

$$F_1^{(1)} = R_A \qquad F_3^{(1)} + F_1^{(2)} = P \qquad F_3^{(2)} = R_C$$

$$\delta\Omega = \begin{Bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \\ \delta u_4 \\ \delta u_5 \end{Bmatrix}^T \left(\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & 0 & 0 \\ K_{31}^{(1)} & K_{32}^{(1)} & (K_{33}^{(1)} + K_{11}^{(2)}) & K_{12}^{(2)} & K_{13}^{(2)} \\ 0 & 0 & K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ 0 & 0 & K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} - \begin{Bmatrix} R_A \\ 0 \\ P \\ 0 \\ R_C \end{Bmatrix} \right)$$

Incorporating the boundary conditions on displacements

- The stiffness matrix in is singular, which reflects the fact that the two element structure can move as a rigid body.

$$u_1 = 0 \qquad \delta u_1 = 0 \qquad u_5 = 0 \qquad \delta u_5 = 0$$

$$\delta\Omega = \begin{Bmatrix} 0 \\ \delta u_2 \\ \delta u_3 \\ \delta u_4 \\ 0 \end{Bmatrix}^T \left(\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & 0 & 0 \\ K_{31}^{(1)} & K_{32}^{(1)} & (K_{33}^{(1)} + K_{11}^{(2)}) & K_{12}^{(2)} & K_{13}^{(2)} \\ 0 & 0 & K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ 0 & 0 & K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ u_4 \\ 0 \end{Bmatrix} - \begin{Bmatrix} R_A \\ 0 \\ P \\ 0 \\ R_C \end{Bmatrix} \right)$$

or

$$\delta\Omega = \begin{Bmatrix} \delta u_2 \\ \delta u_3 \\ \delta u_4 \end{Bmatrix}^T \left(\begin{bmatrix} K_{22}^{(1)} & K_{23}^{(1)} & 0 \\ K_{32}^{(1)} & (K_{33}^{(1)} + K_{11}^{(2)}) & K_{12}^{(2)} \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} - \begin{Bmatrix} 0 \\ P \\ 0 \end{Bmatrix} \right)$$

By principle of minimum potential energy: $\delta\Omega = 0$

$$\begin{bmatrix} K_{22}^{(1)} & K_{23}^{(1)} & 0 \\ K_{32}^{(1)} & K_{33}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ 0 \end{Bmatrix}$$

Reaction Forces:

$$R_A = K_{12}^{(1)} u_2 + K_{13}^{(1)} u_3 \qquad R_C = K_{31}^{(2)} u_3 + K_{32}^{(2)} u_4$$

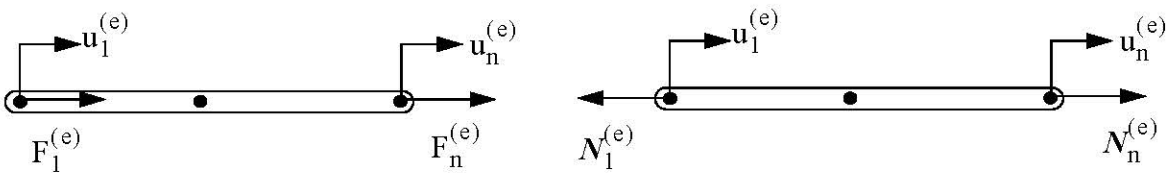
Element strain energy

In Rayleigh-Ritz method we showed that the strain energy is half the work potential of the forces at equilibrium.

$$\boxed{U_A^{(e)} = \frac{1}{2}\sum_{j=1}^{n} u_j^{(e)} R_j^{(e)}}$$

For $p_x = 0$ $\quad R_1^{(e)} = F_1^{(e)} \quad R_n^{(e)} = F_n^{(e)} \quad R_j^{(e)} = 0 \quad j = 2, (n-1)$

$$U_A^{(e)} = \frac{1}{2}(u_1^{(e)} F_1^{(e)} + u_n^{(e)} F_n^{(e)})$$



$F_1^{(e)} = -N_1^{(e)} \qquad F_n^{(e)} = N_n^{(e)}$

For $p_x = 0$ $\quad N_1^{(e)} = N_n^{(e)} = N^{(e)}$

$$U_A^{(e)} = \frac{1}{2}(u_n^{(e)} N_n^{(e)} - u_1^{(e)} N_1^{(e)}) = \frac{1}{2}(u_n^{(e)} - u_1^{(e)}) N^{(e)}$$

Mesh Refinement

- Elements with high strain energy identify the region of the body where mesh should be refined.

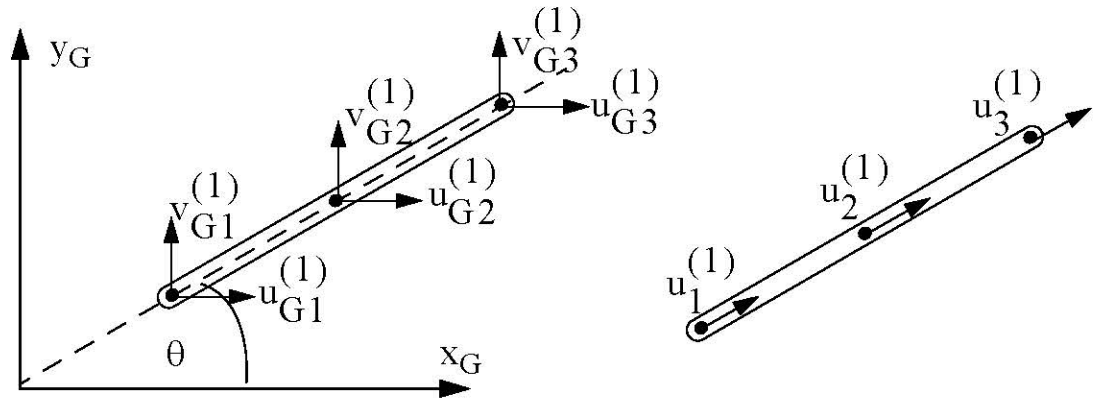
  The *h-method* of mesh refinement reduces the size of element.

  The *p-method* of mesh refinement increases the order of polynomial in an element.

  The *r-method* of mesh refinement relocates the position of a node.

  Combinations: hr-method, hp-method, hpr-method

Coordinate transformation



The displacement vector in global orientation: $\overline{D}_1^{(e)} = u_{G1}^{(e)}\vec{i} + v_{G1}^{(e)}\vec{j}$

The unit vector: $\overline{e_L} = \cos\theta\vec{i} + \sin\theta\vec{j}$

Axial displacement: $u_1^{(1)} = \overline{D}_1^{(e)} \bullet \overline{e_L} = u_{G1}^{(e)}\cos\theta + v_{G1}^{(e)}\sin\theta$

$$\begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \\ u_3^{(e)} \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} u_{G1}^{(e)} \\ v_{G1}^{(e)} \\ u_{G2}^{(e)} \\ v_{G2}^{(e)} \\ u_{G3}^{(e)} \\ v_{G3}^{(e)} \end{Bmatrix} = [T] \begin{Bmatrix} u_{G1}^{(e)} \\ v_{G1}^{(e)} \\ u_{G2}^{(e)} \\ v_{G2}^{(e)} \\ u_{G3}^{(e)} \\ v_{G3}^{(e)} \end{Bmatrix}$$

- The [T] matrix in the above equation is 3 x 6 matrix relating the local and the global coordinate system.
- For n nodes on the element the matrix [T] would be n x 2n size.

Transformation equation in matrix form: $\{u^{(e)}\} = [T]\left\{u_G^{(e)}\right\}$

Potential energy in matrix form:

$$\delta\Omega^{(e)} = \{\delta u^{(e)}\}^T([K^{(e)}]\{u^{(e)}\} - \{R^{(e)}\}) \qquad \text{or}$$

$$\delta\Omega^{(e)} = \left\{\delta u_G^{(e)}\right\}^T [T]^T\left([K^{(e)}][T]\left\{u_G^{(e)}\right\} - \{R^{(e)}\}\right) \text{ or}$$

$$\delta\Omega^{(e)} = \left\{\delta u_G^{(e)}\right\}^T\left([T]^T[K^{(e)}][T]\left\{u_G^{(e)}\right\} - [T]^T\{R^{(e)}\}\right) \text{ or}$$

Define: $[K_G^{(e)}] = [T]^T[K^{(e)}][T] \qquad \left\{R_G^{(e)}\right\} = [T]^T\{R^{(e)}\}$

$$\delta\Omega^{(e)} = \left\{\delta u_G^{(e)}\right\}^T\left([K_G^{(e)}]\left\{u_G^{(e)}\right\} - \left\{R_G^{(e)}\right\}\right)$$

Linear and quadratic axial elements

We assume the following:

---the distributed load p_X is evaluated at the mid point of the element and has a uniform value p_o

---the cross-sectional area (A) is evaluated at the mid point of the element.

---the modulus of elasticity (E) is constant over the element.

Linear:

$$u^{(e)}(x) = u_1^{(e)} L_1(x) + u_2^{(e)} L_2(x)$$

$$L_1(x) = \left(\frac{x - x_2}{x_1 - x_2}\right) \qquad \{u^{(e)}\} = \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix}$$

$$L_2(x) = \left(\frac{x - x_1}{x_2 - x_1}\right)$$

$$\boxed{[K^{(e)}] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \{R^{(e)}\} = \frac{p_o L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} F_1^{(e)} \\ F_2^{(e)} \end{Bmatrix}}$$

Quadratic:

$$u^{(e)}(x) = u_1^{(e)} L_1(x) + u_2^{(e)} L_2(x) + u_3^{(e)} L_3(x)$$

$$L_1(x) = \left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right)$$

$$L_2(x) = \left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_3}{x_2 - x_3}\right) \qquad \{u^{(e)}\} = \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \\ u_3^{(e)} \end{Bmatrix}$$

$$L_3(x) = \left(\frac{x - x_1}{x_3 - x_1}\right)\left(\frac{x - x_2}{x_3 - x_2}\right)$$

$$\boxed{[K^{(e)}] = \frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \qquad \{R\} = \frac{p_o L}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} F_1^{(e)} \\ 0 \\ F_3^{(e)} \end{Bmatrix}}$$

$$\boxed{\delta\Omega^{(e)} = \{\delta u^{(e)}\}^T([K^{(e)}]\{u^{(e)}\} - \{R^{(e)}\})}$$

Steps in FEM procedure

1. Obtain element stiffness and element load vector.
2. Transform from local orientation to global orientation.
3. Assemble the global stiffness matrix and load vector.
4. Incorporate the external loads
5. Incorporate the boundary conditions.
6. Solve the algebraic equations for nodal displacements.
7. Obtain reaction force, stress, internal forces, strain energy.
8. Interpret and check the results.
9. Refine mesh if necessary, and repeat the above steps.

8.8 A rectangular tapered aluminum bar (E_{al} = 10,000 ksi, ν = 0.25) is shown in Figure 8.8. The depth of the tapered section varies as: $h(x) = 4 - 0.04x$. Find the stress at point B, displacement at point C and the strain energy in each element using the following FEM model: one linear element in AB and one quadratic element in BC.

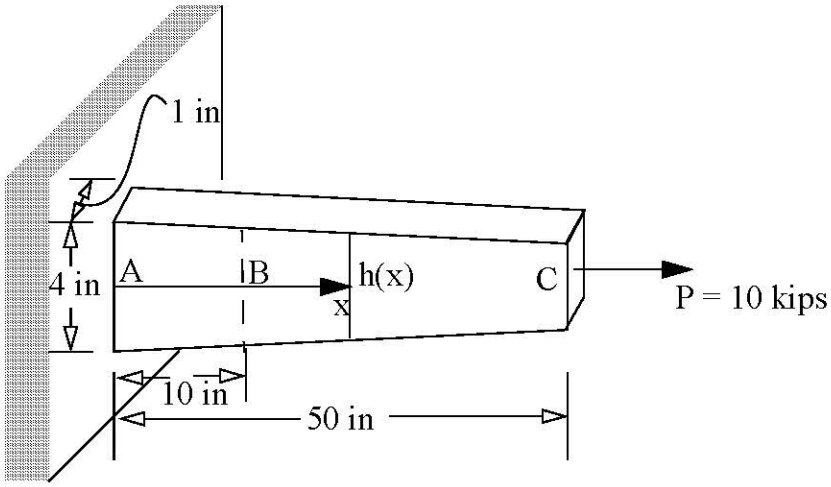


Fig. P8.8

8.10 A force F= 20 kN is applied to the roller that slides inside a slot as shown in Fig. P8.10. Both bars have an area of cross-section of $A = 100\ mm^2$ and a Modulus of Elasticity E = 200 GPa. Bar AP and BP have lengths of L_{AP}= 200 mm and L_{BP}= 250 mm respectively. Determine the displacement of the roller and the reaction force on the roller using linear elements to represent each bar.

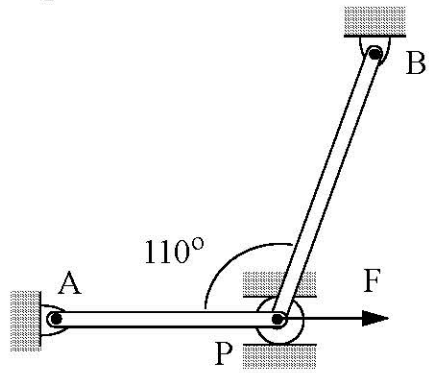


Fig. P8.10

Symmetric beam elements

- In beam bending problems deflection v and its derivative $\frac{dv}{dx}$ must be continuous at all points on the beam including the element ends.
- Lagrange polynomials cannot be used for representing v because it would result in slope being discontinuous at the element end irrespective of the order of polynomial.

Deflection Approximation: $v(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3$

Conditions:

$$v(x_1) = v_1^{(e)} \qquad \frac{dv}{dx}(x_1) = \theta_1^{(e)}$$

$$v(x_2) = v_2^{(e)} \qquad \frac{dv}{dx}(x_2) = \theta_2^{(e)}$$

$$v(x) = f_1(x)v_1^{(e)} + f_2(x)\theta_1^{(e)} + f_3(x)v_2^{(e)} + f_4(x)\theta_2^{(e)}$$

$$f_1(x) = 1 - 3\left(\frac{x - x_1}{L}\right)^2 + 2\left(\frac{x - x_1}{L}\right)^3$$

$$f_2(x) = L\left[\left(\frac{x - x_1}{L}\right) - 2\left(\frac{x - x_1}{L}\right)^2 + \left(\frac{x - x_1}{L}\right)^3\right]$$

$$f_3(x) = 3\left(\frac{x - x_1}{L}\right)^2 - 2\left(\frac{x - x_1}{L}\right)^3$$

$$f_4(x) = L\left[-\left(\frac{x - x_1}{L}\right)^2 + \left(\frac{x - x_1}{L}\right)^3\right]$$

Conditions on interpolating functions

$$v(x_1) = v_1^{(e)} = f_1(x_1)v_1^{(e)} + f_2(x_1)\theta_1^{(e)} + f_3(x_1)v_2^{(e)} + f_4(x_1)\theta_2^{(e)}$$

$$f_1(x_1) = 1 \qquad f_2(x_1) = 0 \qquad f_3(x_1) = 0 \qquad f_4(x_1) = 0$$

$$\frac{dv}{dx}(x_1) = \frac{df_1}{dx}(x_1)v_1^{(e)} + \frac{df_2}{dx}(x_1)\theta_1^{(e)} + \frac{df_3}{dx}(x_1)v_2^{(e)} + \frac{df_4}{dx}(x_1)\theta_2^{(e)}$$

$$= \theta_1^{(e)}$$

$$\frac{df_1}{dx}(x_1) = 0 \qquad \frac{df_2}{dx}(x_1) = 1 \qquad \frac{df_3}{dx}(x_1) = 0 \qquad \frac{df_3}{dx}(x_1) = 0$$

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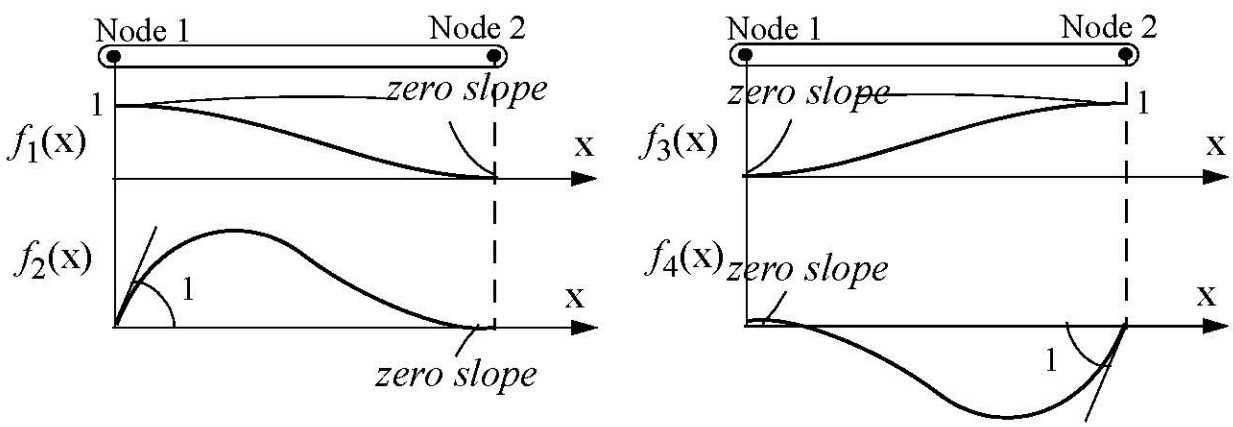
Hermite Polynomials for Beam Bending



Element Stiffness Matrix:

From Rayleigh-Ritz the stiffness matrix is:

$$K_{jk} = \int_0^L (EI_{zz})\left(\frac{d^2 f_j}{dx^2}\right)\left(\frac{d^2 f_k}{dx^2}\right)dx$$

Assume:

- the area moment of inertia (I) is evaluated for the cross-section at the mid point of the element.
- the modulus of elasticity (E) is constant over the element.

$$\{v^{(e)}\} = \begin{Bmatrix} v_1^{(e)} \\ \theta_1^{(e)} \\ v_2^{(e)} \\ \theta_2^{(e)} \end{Bmatrix} \qquad [K^{(e)}] = \frac{2EI}{L^3}\begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix}$$

Element Load Vector

From Rayleigh-Ritz the load vector is:

$$R_j = \int_0^L p_y(x) f_j(x)\ dx + \sum_{q=1}^{m_1} F_q f_j(x_q) + \sum_{q=1}^{m_1} M_q \frac{df_j}{dx}(x_q)$$

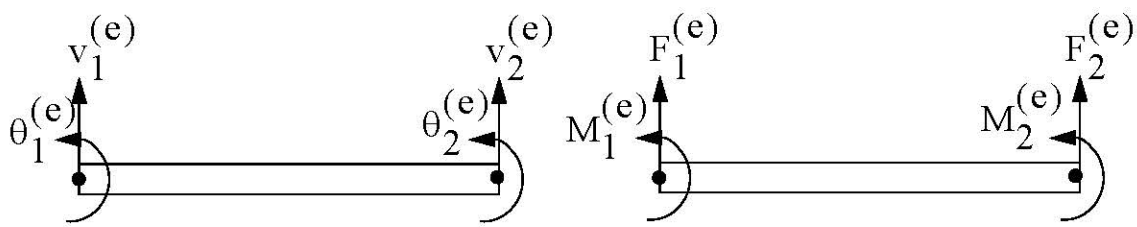
- Assume point forces and moments can only be applied at element end nodes.

$$R_j^{(e)} = \int_0^L p_y(x) f_j(x)\ dx + F_1^{(e)} f_j(x_1) + F_2^{(e)} f_j(x_2) + M_1^{(e)} \frac{df_j}{dx}(x_1) + M_2^{(e)} \frac{df_j}{dx}(x_2)$$

- the distributed load p_y is evaluated at the mid point of the element and has a uniform value p_o.

$$\{R^{(e)}\} = \frac{p_0 L}{12}\begin{Bmatrix} 6 \\ L \\ 6 \\ -L \end{Bmatrix} + \begin{Bmatrix} F_1^{(e)} \\ M_1^{(e)} \\ F_2^{(e)} \\ M_2^{(e)} \end{Bmatrix}$$

Positive directions for displacements, slopes, nodal forces and moments.



$$\delta\Omega_B^{(e)} = \{\delta v^{(e)}\}^T([K^{(e)}]\{v^{(e)}\} - \{R^{(e)}\})$$

8.23 Using a single beam element for AB and a single beam element for BC in Fig. P8.23, determine (a) the slope at B and C (b) reaction force and moment at A.

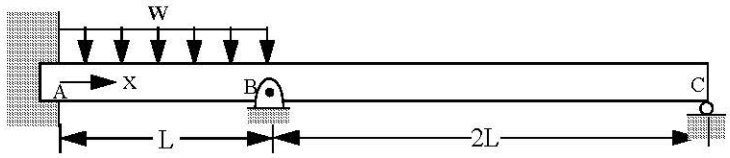


Fig. P8.23