9 Finite Element Method

LEARNING OBJECTIVE

1. To understand the perspective, the key issues, and the terminology of the finite element method.
2. To understand the procedural steps of solving problems by the finite element method.

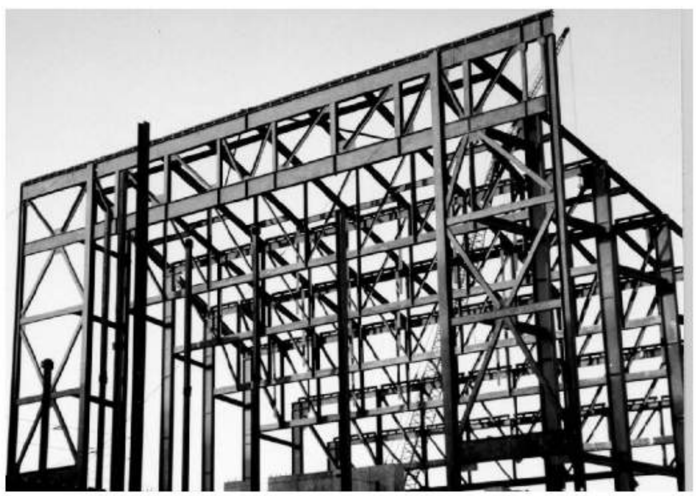

Figure 9.1 Discrete elements of a building frame.

The finite element method is a versatile numerical method that is ubiquitous in stress analysis and in the design of machines and structures. Figure 9.1 shows the frame of a building representing an assembly of beams, columns, and axial members. Associated with each structural element of the building frame is a stiffness matrix, and all these matrices together can be assembled into a global stiffness matrix to represent the structure. This process of assembly is methodically done in the finite element method, as will be seen in this chapter.

The finite element method began as a matrix method of analysis. There are two versions of it: the **stiffness method**, in which the displacements of points on the structure are unknown, and the **flexibility method**, in which the internal forces in the structural members are unknown. Commercial computer programs are usually based on the stiffness method, which is described in this chapter. Such simple elements as axial rods, circular shafts, and symmetric beams will be the primary focus in this introduction to the finite element method. Our treatment of two-dimensional elements (Section 9.6) will be brief.

The set of equations used in the stiffness method are the equilibrium equations relating the displacement of points. The Rayleigh-Ritz method, which predates the finite element method, is a formal procedure for deriving equilibrium equations in matrix form, as was seen in Section 7.9. From a theoretical viewpoint, the primary difference between the Rayleigh-Ritz method and the finite element method is that the kinematically admissible functions used in finding the approximate solutions are defined over the entire structural member in the Rayleigh-Ritz method, while in the finite element method the functions are piecewise kinematically admissible. The small theoretical difference, however, results in a dramatically different perspective of solving problems by the finite element method

Many of the conclusions and equations of the Rayleigh-Ritz method are applicable to the finite element method. We will briefly study "Lagrange polynomials," which are used pervasively in the finite element method as the piecewise kinematically admissible displacement functions. The key steps of the finite element method will be elaborated for simple structures made from axial rods, circular shafts, and symmetric beams.

9.1 TERMINOLOGY

The finite element method (FEM) originated in structures as a matrix method. Each row of the matrix represents an equilibrium equation. In the matrix method, the equilibrium equations were obtained by force and moment balance. In this chapter we will obtain the equilibrium equations by minimizing the potential energy. Thus, the finite element method formulation presented here is very similar to the Rayleigh-Ritz method with one important difference: the kinematically admissible displacement functions in the finite element method are defined piecewise continuously over small (finite) domains; these are the **elements**. The boundary points of the elements are called nodes, although nodes can also be points inside the element. The constants multiplying the piecewise kinematically admissible functions are the displacements of the nodes. Thus, **nodes** are points on the structure at which displacements and rotations are to be found or prescribed. The kinematically admissible functions are called **interpolation functions** because they can be used to interpolate the values of displacements between the nodes. The representation of a structure by elements and nodes is called a **mesh**. A mesh with boundary conditions, applied loads, and material property is called a **model**. A model is a finite element representation of a real-life problem, and the accuracy of the model's predictions is determined by the assumptions and limitations that are made in constructing the finite element model and the errors introduced in solving the model by numerical methods.

The use of piecewise kinematically admissible functions changes the perspective with which we view and solve a problem by means of the finite element method. To elaborate this perspective, consider the statement of minimum potential energy in Equation (7.31). A structure could be made up of axial members, circular shafts, symmetric beams, and other members such as curved beams, plates, and shells. Potential energy is a scalar quantity and can be written as the sum of the potential energies of all the structural members ($\delta\Omega^{(i)}$). Equation (7.31) can be written as:

$$\delta\Omega = \sum_{i=1}^{n} \delta\Omega^{(i)} \tag{9.1}$$

Equation (9.1) is valid for structural members of all types, irrespective of orientation. We could thus develop the potential energy in matrix form for each member separately. That is, we could develop matrices at the element level in a local coordinate system without regard to how a member is used in the structure. The individual local matrices, called **element stiffness matrices**, could be assembled by using Equation (9.1) to form the **global stiffness matrix** of the entire structure. This perspective of reducing the *complexity of analyzing large structures to the analysis of simple individual members* (*elements*) is what makes the finite element method such a versatile and popular tool in structural (and engineering application) analysis.

We will develop element stiffness matrices for axial members, circular shafts, and symmetric beams. We will use these matrices to analyze simple structures to elaborate the principles of assembly represented by Equation (9.1). The analysis of complex structure requires the use of computers. The FEM is now a very mature technique, and there are many commercially available software packages that can be used for solving engineering problems.

9.2 LAGRANGE POLYNOMIALS

Lagrange polynomials were discovered independently of the finite element method. These functions are used for interpolations of many quantities. The polynomials are introduced by using the axial members to provide the motivation and practical relevance of these functions in the FEM.

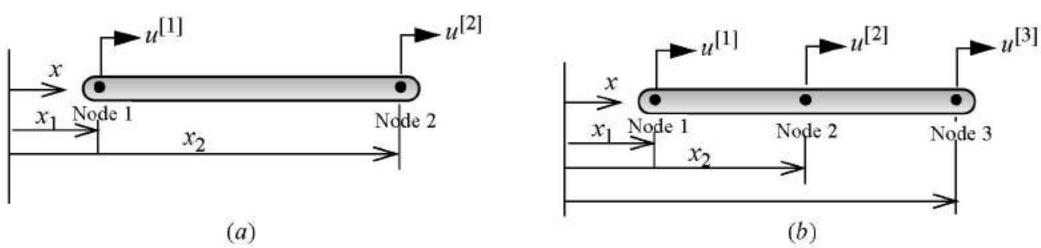

Figure 9.2 Axial (*a*) linear and (*b*) quadratic elements.

Figure 9.2*a* shows an axial member (element) with two nodes. The axial displacements u_1 and u_2 are the degrees of freedom (generalized displacements) in terms of which we plan to derive the element stiffness matrix.The element has 2 degrees of freedom, and so we choose a linear function $u(x) = C_1 + C_2 x$ with two unknown parameters. We note that at $x = x_1$ the displacement $u(x_1) = u_1$, and at $x = x_2$ the displacement $u(x_2) = u_2$. The constants C_1 and C_2 can be solved in terms of u_1 and u_2 and substituted into the linear function representation to obtain Equation (9.2a),

$$u(x) = u_1\left(\frac{x - x_2}{x_1 - x_2}\right) + u_2\left(\frac{x - x_1}{x_2 - x_1}\right) = u_1 \mathcal{L}_1(x) + u_2 \mathcal{L}_2(x) = \sum_{i=1}^{2} u_i \mathcal{L}_i(x) \tag{9.2a}$$

where

$$\mathcal{L}_1(x) = \left(\frac{x - x_2}{x_1 - x_2}\right) \quad \text{and} \quad \mathcal{L}_2(x) = \left(\frac{x - x_1}{x_2 - x_1}\right) \tag{9.2b}$$

A linear representation of displacement is sufficient if the forces are applied only at the element end and only if the cross-sectional area does not change across the element. If the axial member has a distributed load, or if the member is tapered, then the axial displacement is no longer linear inside the element. A quadratic or higher-order polynomial may converge to the actual solution faster than a linear element. Figure 9.2*b* shows an element with three nodes. With 3 degrees of freedom, we can start with a quadratic displacement function $u(x) = C_1 + C_2x + C_3x^2$, solve the constant in terms of the nodal displacement, and obtain an equation analogous to Equation (9.2a). This process would be tedious for higher-order polynomials. So we use an alternative approach. We represent the displacement in the element by Equation (9.3a),

$$u(x) = \sum_{i=1}^{n} u_i \mathcal{L}_i(x) \tag{9.3a}$$

where n is the degrees of freedom (number of nodes, in this case) of the element that can be used for representing the $(n-1)$ order of polynomials. Now at the jth node, the displacement $u(x_j) = u_j$, and we obtain Equation (9.3b).

$$u(x_j) = \sum_{i=1}^{n} u_i \mathcal{L}_i(x_j) = u_j \tag{9.3b}$$

For Equation (9.3b) to be true, the property given in Equation (9.4a) must hold.

$$\mathcal{L}_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{9.4a}$$

Equation (9.4a) implies that the polynomials $\mathcal{L}_i(x)$ are such that the value is 1 on its own ith node and zero at other nodes. Figure 9.3 shows the approximate plots for linear and quadratic $\mathcal{L}_i(x)$ that meet this requirement.

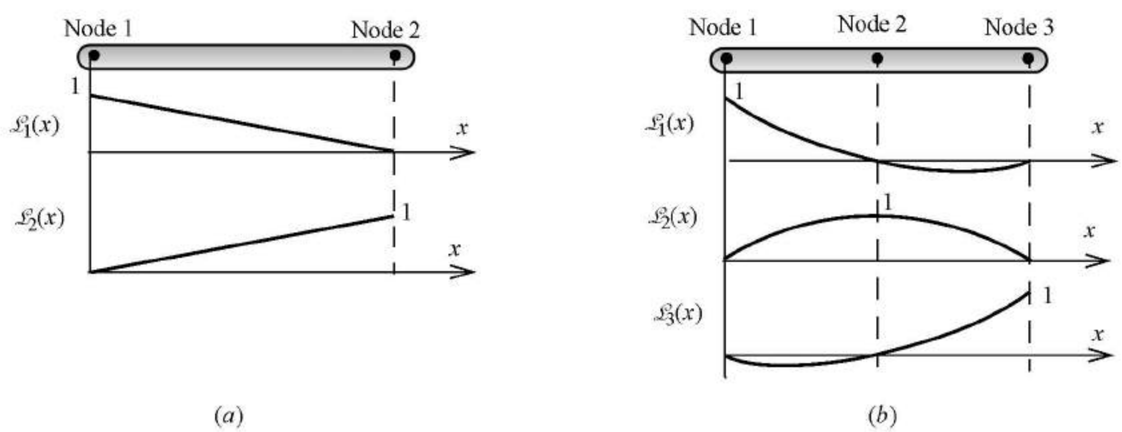

Figure 9.3 (*a*) Linear and (*b*) quadratic Lagrange polynomials.

Consider now $\mathcal{L}_1$ for the quadratic. If we represent $\mathcal{L}_1(x) = a_1(x - x_2)(x - x_3)$, its value is zero at nodes 2 and 3. We can now determine the constant a_1 such that $\mathcal{L}_1$ at node 1 is equal to one, and we obtain Equation (9.5a).

$$\mathcal{L}_1(x) = \left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right) \tag{9.5a}$$

In a similar manner, we can start with $\mathcal{L}_2(x) = a_2(x - x_3)(x - x_1)$ and $\mathcal{L}_3(x) = a_3(x - x_1)(x - x_2)$ and determine the value of a_2 and a_3 such that $\mathcal{L}_2$ and $\mathcal{L}_3$ at nodes 2 and 3, respectively, have a value of one to obtain Equation (9.5b).

$$\mathcal{L}_2(x) = \left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_3}{x_2 - x_3}\right) \quad \text{and} \quad \mathcal{L}_3(x) = \left(\frac{x - x_1}{x_3 - x_1}\right)\left(\frac{x - x_2}{x_3 - x_2}\right) \tag{9.5b}$$

The process we used to obtain the polynomials for the quadratic can now be generalized to obtain Equation (9.6),

$$\mathcal{L}_i(x) = \prod_{\substack{j=1 \\ i \neq j}}^{n} \left[\frac{(x - x_j)}{(x_i - x_j)}\right] \tag{9.6}$$

where $\Pi_{j=1}^{n}[\cdots]$ represents the product of the terms in square brackets. The functions defined by Equation (9.6) are called *Lagrange polynomials*.

Functions represented by Lagrange polynomials will be continuous at the element ends. Inside the elements, all orders of derivatives are defined as polynomials are continuous. However, the continuity of the derivative of the function cannot be ensured at the element end irrespective of the order of polynomials when Lagrange polynomials are used for representing the function. Figure 9.4 shows a possible variation of a displacement field represented by Lagrange polynomials over two adjoining elements. Displacements at nodes are independent parameters that can have any value; hence the variation shown in Figure 9.4 is a possibility. As can be seen from Figure 9.4, the continuity of the displacement is maintained, but its first derivative is not continuous for either the linear or the quadratic element at the element end node.

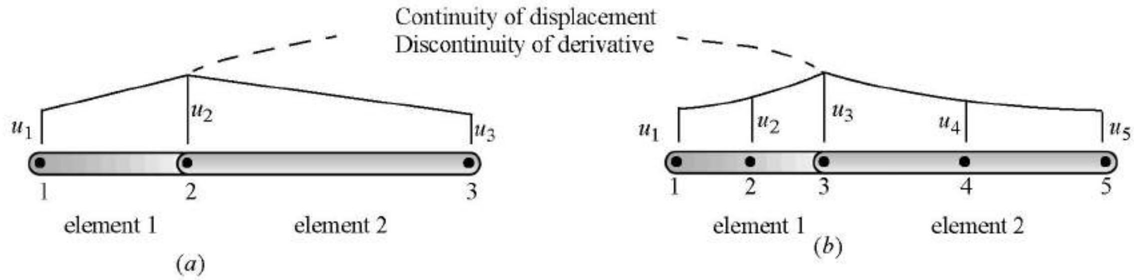

Figure 9.4 Possible variation of displacement for (*a*) linear and (*b*) quadratic elements.

9.3 AXIAL ELEMENTS

The strains, stresses, and internal forces on any element *e* can be obtained once the nodal displacements in Equation (9.3a) have been determined. To differentiate quantities at the element and global levels, we will use a superscript to designate quantities at the element level and no superscript at the global level. Thus $u_3^{(1)}$ and $u_3^{(2)}$ will refer to the displacement of node 3 for element 1 and 2, respectively, while u_3 refers to the displacement of node 3 on the actual structure. The strains, stresses, and internal force can be written as follows:

$$\varepsilon_{xx}^{(e)} = \frac{\mathrm{d}u^{(e)}}{\mathrm{d}x} = \sum_{i=1}^{n} u_i^{(e)} \frac{\mathrm{d}\mathcal{L}_i}{\mathrm{d}x} \tag{9.7a}$$

$$\sigma_{xx}^{(e)} = E^{(e)} \varepsilon_{xx}^{(e)} = E^{(e)} \sum_{i=1}^{n} u_i^{(e)} \frac{\mathrm{d}\mathcal{L}_i}{\mathrm{d}x} \tag{9.7b}$$

$$N^{(e)} = A^{(e)} \sigma_{xx}^{(e)} = E^{(e)} A^{(e)} \sum_{i=1}^{n} u_i^{(e)} \frac{\mathrm{d}\mathcal{L}_i}{\mathrm{d}x} \tag{9.7c}$$

The element stiffness matrix and the element load vector can be obtained by comparing Equation (9.3a) and Equation (7.32a) of Rayleigh-Ritz method. We note the generalized displacements $C_i = u_i^{(e)}$ and the kinematically admissible functions $f_i(x) = \mathcal{L}_i(x)$.

9.3.1 Element Stiffness Matrix

Substituting $f_i(x) = \mathcal{L}_i(x)$ into Equation (7.33d), we obtain the element stiffness matrix as shown in Equation (9.8a).

$$\boxed{K_{jk}^{(e)} = \int_0^L E^{(e)} A^{(e)} \left(\frac{\mathrm{d}\mathcal{L}_j}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}\mathcal{L}_k}{\mathrm{d}x}\right) dx} \tag{9.8a}$$

9.3.2 Element Load Vector

To obtain the element load vector, we assume that point forces can be applied only at the element end nodes. This requirement is easily met during mesh creation, where we create an element such that the point forces are at the end. From this requirement and from Equation (7.33g), we obtain Equation (9.9a).

$$\boxed{R_j^{(e)} = \int_0^L p_x(x) \mathcal{L}_j(x) dx + F_1^{(e)} \mathcal{L}_j(x_1) + F_n^{(e)} \mathcal{L}_j(x_n)} \tag{9.9a}$$

We know from Equation (9.4a) that $\mathcal{L}_j(x_1)$ is zero except when $j = 1$, and $\mathcal{L}_j(x_n)$ is zero except when $j = n$. Thus, if p_x is zero, only the forces at the end point are nonzero, in accordance with our requirement that external forces be applied at the element end.

For $p_x = 0$:

$$R_1^{(e)} = F_1^{(e)} \qquad R_n^{(e)} = F_n^{(e)} \qquad R_j^{(e)} = 0 \qquad j = 2 \text{ to } (n-1) \tag{9.10a}$$

It should be noted that the force $F_j^{(e)}$ is positive in the positive direction $u_j^{(e)}$. This property will be important during assembly.

9.3.3 Assembly of Global Matrix and Global Load Vector

To assemble a global matrix, we must use Equation (9.1). The assembly process is primarily a careful bookkeeping effort to ensure that the matrix components at the element level add to the correct components in the global matrix. The governing criterion is that the displacement function at the node where two elements meet be continuous. The assembly process is elaborated by using two quadratic axial elements to model a simple structure, as shown in Figure 9.5. We will assume that there is no distributed force (i.e., $p_x = 0$).

The virtual variation in potential energy due to the virtual displacement in an element can be written as in Equation (9.11a).

$$\delta\Omega^{(m)} = \sum_{i=1}^{n} \frac{\partial \Omega^{(m)}}{\partial u_i^{(m)}} \delta u_i^{(m)} \tag{9.11a}$$

In Equation (7.35b) we evaluated the derivative of potential energy with respect to C_i. Following steps similar to those in Equations (7.35b) through (7.35e), we obtain the derivative of the element's potential energy, which we substitute in Equation (9.11a) to obtain Equation (9.12a).

$$\delta\Omega^{(m)} = \sum_{i=1}^{n} \delta u_i^{(m)} \sum_{j=1}^{n} (K_{ij}^{(m)} u_j^{(m)} - R_i^{(m)}) \tag{9.12a}$$

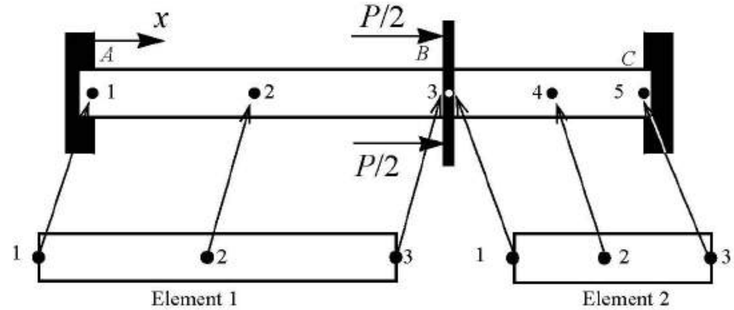

Figure 9.5 Assembly of two quadratic axial elements.

Equation (9.12a) can be written in matrix form for the two elements shown in Figure 9.5 as shown in Equation (9.13a) and (9.13b).

$$\delta\Omega^{(1)} = \begin{Bmatrix} \delta u_1^{(1)} \\ \delta u_2^{(1)} \\ \delta u_3^{(1)} \end{Bmatrix}^T \left(\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{Bmatrix} - \begin{Bmatrix} F_1^{(1)} \\ 0 \\ F_3^{(1)} \end{Bmatrix} \right) \tag{9.13a}$$

$$\delta\Omega^{(2)} = \begin{Bmatrix} \delta u_1^{(2)} \\ \delta u_2^{(2)} \\ \delta u_3^{(2)} \end{Bmatrix}^T \left(\begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} \begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_3^{(2)} \end{Bmatrix} - \begin{Bmatrix} F_1^{(2)} \\ 0 \\ F_3^{(2)} \end{Bmatrix} \right) \tag{9.13b}$$

From Figure 9.5 we note the following relationship between the displacements at the element nodes to the displacements on the original structure.

$$\begin{array}{lll} u_1^{(1)} = u_1 & u_2^{(1)} = u_2 & u_3^{(1)} = u_3 \\ u_1^{(2)} = u_3 & u_2^{(2)} = u_4 & u_3^{(2)} = u_5 \end{array} \tag{9.14a}$$

We note that there are five nodes on the actual structure. Thus the stiffness matrix for the potential energy of the entire structure will have 5 rows and 5 columns. We can use Equation (9.14a) to write Equation (9.13a) and (9.13b) as Equation (9.14b) and (9.14c).

$$\delta\Omega^{(1)} = \begin{Bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \\ \delta u_4 \\ \delta u_5 \end{Bmatrix}^T \left(\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & 0 & 0 \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} - \begin{Bmatrix} F_1^{(1)} \\ 0 \\ F_3^{(1)} \\ 0 \\ 0 \end{Bmatrix} \right) \tag{9.14b}$$

$$\delta\Omega^{(2)} = \begin{Bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \\ \delta u_4 \\ \delta u_5 \end{Bmatrix}^T \left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} \\ 0 & 0 & K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ 0 & 0 & K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ F_1^{(2)} \\ 0 \\ F_3^{(2)} \end{Bmatrix} \right) \qquad \textbf{(9.14c)}$$

From Equation (9.1) we know that the total potential energy of the structure is the sum of potential energies of all the elements (i.e., $\delta\Omega = \delta\Omega^{(1)} + \delta\Omega^{(2)}$). We thus obtain Equation (9.14d).

$$\delta\Omega = \begin{Bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \\ \delta u_4 \\ \delta u_5 \end{Bmatrix}^T \left(\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & 0 & 0 \\ K_{31}^{(1)} & K_{32}^{(1)} & (K_{33}^{(1)} + K_{11}^{(2)}) & K_{12}^{(2)} & K_{13}^{(2)} \\ 0 & 0 & K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ 0 & 0 & K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} - \begin{Bmatrix} F_1^{(1)} \\ 0 \\ F_3^{(1)} + F_1^{(2)} \\ 0 \\ F_3^{(2)} \end{Bmatrix} \right) \qquad \textbf{(9.14d)}$$

Equation (9.14d) highlights that the element stiffness matrix and the element load components that are added correspond to the degrees of freedom associated with the shared node of the elements.

9.3.4 Incorporating the External Concentrated Forces

From Figure 9.5 we see that at nodes 1 and 5 there will be reaction forces at supports A and C, which we label R_A and R_C. Force $F_3^{(1)}$ is in the direction of $u_3^{(1)}$, force $F_1^{(2)}$ is in the direction of $u_1^{(2)}$, and the applied force is in the direction of u_3. Since $u_3^{(1)}$, $u_1^{(2)}$, and u_3 are equal, we see that the applied force P is equal to the sum of the two forces applied at the element nodes. We thus have the equivalence relationship of Equation (9.15a).

$$F_1^{(1)} = R_A \qquad F_3^{(1)} + F_1^{(2)} = P \qquad F_3^{(2)} = R_C \qquad \textbf{(9.15a)}$$

Substituting Equation (9.15a) into Equation (9.14d), we obtain Equation (9.15b).

$$\delta\Omega = \begin{Bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \\ \delta u_4 \\ \delta u_5 \end{Bmatrix}^T \left(\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & 0 & 0 \\ K_{31}^{(1)} & K_{32}^{(1)} & (K_{33}^{(1)} + K_{11}^{(2)}) & K_{12}^{(2)} & K_{13}^{(2)} \\ 0 & 0 & K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ 0 & 0 & K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} - \begin{Bmatrix} R_A \\ 0 \\ P \\ 0 \\ R_C \end{Bmatrix} \right) \qquad \textbf{(9.15b)}$$

9.3.5 Incorporating the Boundary Conditions on Displacements

The stiffness matrix in Equation (9.15b) is singular (i.e., its determinant is zero). The singular nature of the matrix reflects the fact that the two element structures can move as a rigid body. To eliminate this rigid body mode, we impose the boundary conditions of zero displacement at nodes 1 and 5 shown in Equation (9.16a).

$$u_1 = 0 \qquad \delta u_1 = 0 \qquad u_5 = 0 \qquad \delta u_5 = 0 \qquad \textbf{(9.16a)}$$

Substituting Equation (9.16a) in Equation (9.15b), we obtain Equation (9.16b). Note that the zero values of displacement caused all the components of the matrix in the corresponding row to be multiplied by zero, thus eliminating rows 1 and 5. Also, the zero value in the variation of the displacement caused all the components of the matrix in the corresponding columns to be multiplied by zero, thus eliminating columns 1 and 5. By comparing Equation (9.15b) and (9.16b), we observe that rows and column corresponding to nodes 1 and 5 are indeed eliminated.

$$\delta\Omega = \begin{Bmatrix} \delta u_2 \\ \delta u_3 \\ \delta u_4 \end{Bmatrix}^T \left(\begin{bmatrix} K_{22}^{(1)} & K_{23}^{(1)} & 0 \\ K_{32}^{(1)} & (K_{33}^{(1)} + K_{11}^{(2)}) & K_{12}^{(2)} \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} - \begin{Bmatrix} 0 \\ P \\ 0 \end{Bmatrix} \right) \qquad \textbf{(9.16b)}$$

By the principle of minimum potential energy, the virtual variation of the potential energy due to virtual displacement must be zero (i.e., $\delta\Omega = 0$). Since the virtual displacement cannot be zero, the remaining terms in the bracket must be zero. We obtain the set of algebraic equations shown in Equation (9.16c).

$$\begin{bmatrix} K_{22}^{(1)} & K_{23}^{(1)} & 0 \\ K_{32}^{(1)} & K_{33}^{(1)}+K_{11}^{(2)} & K_{12}^{(2)} \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ 0 \end{Bmatrix} \tag{9.16c}$$

The set of equations in Equation (9.16c) can be solved to obtain the displacements u_2, u_3, and u_4. Thus, we now know the displacements at all nodes. The strains, stresses, and internal forces can be found in each element by using Equation (9.7a), (9.7b), and (9.7c). The reaction forces R_A and R_C can be found from the equations corresponding to first and fifth rows in Equation (9.15b). These equations can be written as shown in Equation (9.16d).

$$R_A = K_{12}^{(1)}u_2 + K_{13}^{(1)}u_3 \qquad \text{and} \qquad R_C = K_{31}^{(2)}u_3 + K_{32}^{(2)}u_4 \tag{9.16d}$$

9.3.6 Element Strain Energy

Equation (7.36) showed that the strain energy is half the work potential of the forces at equilibrium. We note that the C_j in Equation (7.36) are the element nodal displacements. The element strain energy can be written as Equation (9.17a).

$$U_A^{(e)} = \frac{1}{2}\sum_{j=1}^{n} u_j^{(e)}R_j^{(e)} \tag{9.17a}$$

In the absence of distributed forces, we can substitute Equation (9.9a) into Equation (9.17a) to obtain Equation (9.18a).

$$U_A^{(e)} = \frac{1}{2}(u_1^{(e)}F_1^{(e)} + u_n^{(e)}F_n^{(e)}) \tag{9.18a}$$

Figure 9.6 shows relationship between the external and internal nodal forces. We note that $F_1^{(e)} = -N_1^{(e)}$ and $F_n^{(e)} = N_n^{(e)}$. In the absence of distributed forces, the internal force in the element is constant: $N_1^{(e)} = N_n^{(e)} = N^{(e)}$. Substituting these relationship into Equation (9.18a), we obtain Equation (9.19a).

$$U_A^{(e)} = \frac{1}{2}(u_n^{(e)}N_n^{(e)} - u_1^{(e)}N_1^{(e)}) = \frac{1}{2}(u_n^{(e)} - u_1^{(e)})N^{(e)} \tag{9.19a}$$

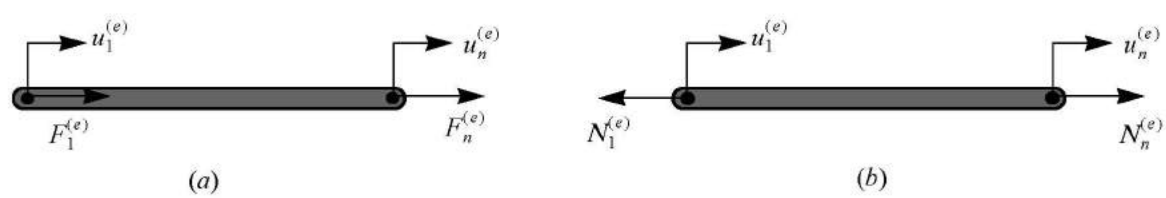

Figure 9.6 (a) External and (b) internal nodal forces.

The strain energy values are used for deciding whether the FEM model needs to be improved. In regions of very large stress gradients, the elements will be highly stressed, and hence elements in these regions will have large strain energies. We would like to refine the mesh so that we can get good resolution of the stress gradients. Elements with high strain energy identify the region of the body in which the mesh should be refined.

There are three major techniques of mesh refinement based on the observation that the higher the number of degrees of freedom, the lower the potential energy, and the better the solution. The *h-method* of mesh refinement reduces the size of the element. The *p-method* of mesh refinement increases the order of polynomials in an element. The *r-method* of mesh refinement relocates the position of a node. There are also combinations such as the *hr-* and *hp*-methods of mesh refinement. These mesh refinement methods of reducing the element size, increasing the polynomial order, and reallocating the nodes are used in regions containing elements with large strain energies.

9.3.7 Transformation Matrix

In the preceding discussion, the orientation of the coordinate system was the same at the element and global levels. In general, axial members are at various orientations in a truss. Thus the element stiffness matrix must be transformed before assembly to correspond to the degrees of freedom in the global coordinate system.

Figure 9.7 shows the global coordinate and the local coordinate, which is along the axial direction of the member. The local coordinate is oriented at an angle θ to the global coordinate. In global coordinates a node point displaces in two directions. The displacement vector of node 1 can be written as $\overline{\mathbf{D}_1} = u_{G1}^{(1)}\overline{\mathbf{i}} + v_{G1}^{(1)}\overline{\mathbf{j}}$, where $\overline{\mathbf{i}}$ and $\overline{\mathbf{j}}$ are the unit vectors in the global x and y directions. The unit vector along the local coordinate can be written as: $\overline{\mathbf{e}_L} = \cos\theta\,\overline{\mathbf{i}} + \sin\theta\,\overline{\mathbf{j}}$. The displacement u_1 in the local coordinate is the component of the vector $\overline{\mathbf{D}_1}$ in the local direction and can be obtained by a dot product as shown in Equation (9.20a).

$$u_1^{(1)} = \overline{\mathbf{D}_1}\cdot\overline{\mathbf{e}_L} = u_{G1}^{(1)}\cos\theta + v_{G1}^{(1)}\sin\theta \tag{9.20a}$$

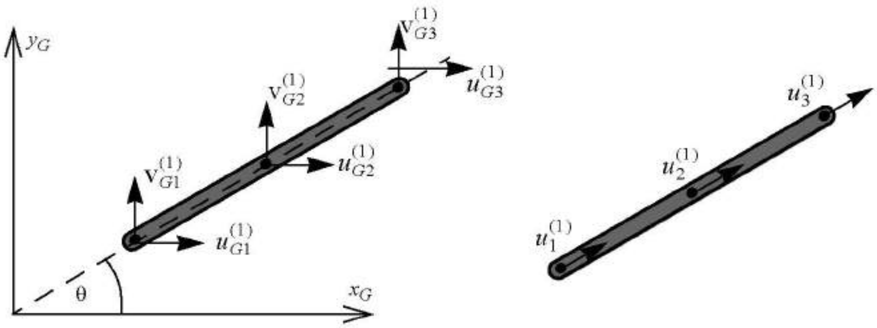

Figure 9.7 Coordinate transformation.

The displacements at other nodes also transform as in Equation (9.20a). The equations transforming the local displacement into global displacements are written in matrix form as shown in Equation (9.20b).

$$\begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} u_{G1}^{(1)} \\ v_{G1}^{(1)} \\ u_{G2}^{(1)} \\ v_{G2}^{(1)} \\ u_{G3}^{(1)} \\ v_{G3}^{(1)} \end{Bmatrix} = [T] \begin{Bmatrix} u_{G1}^{(1)} \\ v_{G1}^{(1)} \\ u_{G2}^{(1)} \\ v_{G2}^{(1)} \\ u_{G3}^{(1)} \\ v_{G3}^{(1)} \end{Bmatrix} \tag{9.20b}$$

The $[T]$ matrix in Equation (9.20b) is 3×6 matrix relating the local and the global coordinate systems. If we had n nodes on the element, then the size of the matrix $[T]$ would be $n \times 2n$. We now rewrite Equation (9.20b) in compact matrix form:

$$\{u^{(1)}\} = [T]\{u_G^{(1)}\} \tag{9.20c}$$

We can also write Equation (9.13a) in compact matrix form, as follows:

$$\delta\Omega^{(1)} = \{\delta u^{(1)}\}^T([K^{(1)}]\{u^{(1)}\} - \{F_1^{(1)}\}) \tag{9.20d}$$

Substituting Equation (9.20c) into Equation (9.20d), we obtain Equation (9.20e).

$$\delta\Omega^{(1)} = \{\delta u_G^{(1)}\}^T[T]^T([K^{(1)}][T]\{u_G^{(1)}\} - \{F_1^{(1)}\}) \qquad \text{or}$$

$$\delta\Omega^{(1)} = \{\delta u_G^{(1)}\}^T([T]^T[K^{(1)}][T]\{u_G^{(1)}\} - [T]^T\{F_1^{(1)}\}) \tag{9.20e}$$

We define the quantities in Equation (9.20f).

$$[K_G^{(1)}] = [T]^T[K^{(1)}][T] \qquad \{F_{G1}^{(1)}\} = [T]^T\{F_1^{(1)}\} \tag{9.20f}$$

Equation (9.20f) use the transformation matrix to transform the stiffness and the load vector. Substituting these equations into Equation (9.20e), we obtain Equation (9.20g).

$$\delta\Omega^{(1)} = \{\delta u_G^{(1)}\}^T([K_G^{(1)}]\{u_G^{(1)}\} - \{F_{G1}^{(1)}\}) \tag{9.20g}$$

Equation (9.20g) has the same form as Equation (9.20d), but now the nodal displacements and forces are in the global coordinate system. Equation (9.20g) can now be used as before in the assembly process.

9.3.8 Linear and Quadratic Axial Elements

In writing the element stiffness matrix and the element load vector, we will assume the following to account for the possibility of varying distributed loads, cross-sectional areas, and moduli of elasticity.

- The distributed load p_x is evaluated at the midpoint of the element and has a uniform value p_0.
- The cross-sectional area A is evaluated at the midpoint of the element.
- The modulus of elasticity E is constant over the element.

The stiffness matrix and the load vector can be calculated by using Equation (9.8a) and (9.9a) and are as given in Equation (9.21a).

$$\{u^{(e)}\} = \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix} \quad [K^{(e)}] = \frac{E^{(e)}A^{(e)}}{L^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \{R^{(e)}\} = \frac{p_0^{(e)}L^{(e)}}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} F_1^{(e)} \\ F_2^{(e)} \end{Bmatrix} \tag{9.21a}$$

We assume that the three nodes of the quadratic element are equally spaced. The stiffness matrix and the load vector, which can be calculated by using Equation (9.8a) and (9.9a), are given in Equation (9.21b).

$$\{u^{(e)}\} = \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \\ u_3^{(e)} \end{Bmatrix} \qquad [K^{(e)}] = \frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \qquad \{R\} = \frac{p_0 L}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} F_1^{(e)} \\ 0 \\ F_3^{(e)} \end{Bmatrix} \qquad \textbf{(9.21b)}$$

The potential energy of the element in compact matrix form can be written as in Equation (9.21c).

$$\delta\Omega^{(e)} = \{\delta u^{(e)}\}^T([K^{(e)}]\{u^{(e)}\} - \{R^{(e)}\}) \qquad \textbf{(9.21c)}$$

9.3.9 Procedural Steps in the Finite Element Method

The analysis of a structure by the FEM proceeds in steps that are generic and can be used for structural elements other than axial rods. However we use axial rods to elaborate the steps.

Step 1. Obtain the element stiffness matrices and the element load vectors.

From the geometry, material properties, and distributed loads, the element stiffness matrices and the element load vectors can be written by using forms equivalent to Equation (9.21a) [or Equation (9.21b)].

Step 2. Transform from local orientation to global orientation.

Equation (9.20f) can be used to transform the element stiffness and the element load vector to the orientation of the global coordinate system.

Step 3. Assemble the global stiffness matrix and load vector.

The element nodal generalized displacements are related to the nodal generalized displacements on the structure to ensure continuity of the generalized displacements. For axial elements, the generalized displacements are just the displacement. As shall be seen, however, for beam elements the generalized displacement also include slopes (i.e., derivatives of displacements).

Step 4. Incorporate the external loads.

The sum of the nodal forces at the element end are replaced by the equivalent external forces applied to the structure.

Step 5. Incorporate the boundary conditions.

Any zero values of the generalized displacement are incorporated by eliminating the corresponding row and column from the global stiffness matrix and load vector.

Step 6. Solve the algebraic equations for the nodal displacements.

Step 7. Obtain the reaction forces, stresses, internal forces, and strain energy.

Step 8. Interpret and check the results.

Step 9. Refine the mesh if necessary, and repeat steps 1 through 8.

EXAMPLE 9.1

A rectangular tapered aluminum bar ($E_{Al} = 10{,}000$ ksi, $\nu = 0.25$) is shown in Figure 9.8. The depth in the tapered section varies as $h(x) = 4 - 0.04x$. Use the following finite element models to solve the problem.

Model 1: two linear elements AB and BC

Model 2: two equal length linear elements in BC and a linear element in AB

For the two models, find the stress at point B, the displacement at point C, and the strain energy in each element. Compare the results with analytical values and comment.

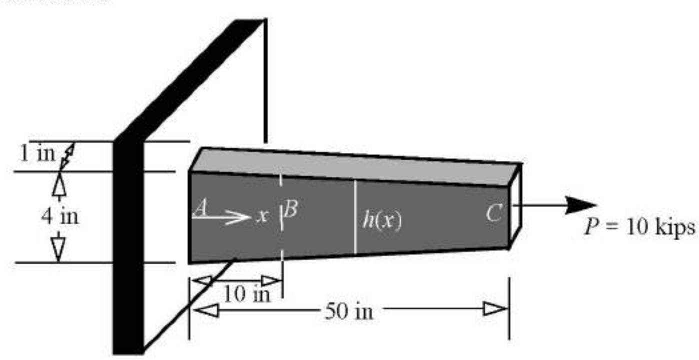

Figure 9.8 Axial member for Example 9.1.

PLAN

We can determine the cross-sectional area as a function of x, and we can determine the cross-sectional area in the middle of each element for determining the element stiffness matrix. For each model, we will follow the steps outlined in Section 9.3.9.

SOLUTION

The cross-sectional area varies as shown in Equation (E1).

$$A(x) = (1)h(x) = 4 - 0.04x \tag{E1}$$

Figure 9.9 shows the two FEM models. In each model, the cross-sectional area is evaluated from Equation (E1) at the midpoint of the element.

Figure 9.9 The FEM models for Example 9.1. (*a*) model 1 and (*b*) model 2.

Model 1

Step 1. The cross-sectional areas at the midpoints for the two elements shown in Figure 9.9*a* are as follows:

$$A_1 = A(5) = 3.8 \text{ in}^2 \quad \text{and} \quad A_2 = A(30) = 2.8 \text{ in}^2 \tag{E2}$$

Noting that $L_1 = 10$ and $L_2 = 40$, we obtain Equation (E3)

$$E_1A_1/L_1 = 3800 \quad \text{and} \quad E_2A_2/L_2 = 700 \tag{E3}$$

The element stiffness matrix and the element load vector for the two elements can be written by using Equation (9.21a) as shown in Equation (E4) and (E5).

$$[K^{(1)}] = \begin{bmatrix} 3800 & -3800 \\ -3800 & 3800 \end{bmatrix} \qquad \{R^{(1)}\} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \end{Bmatrix} \tag{E4}$$

$$[K^{(2)}] = \begin{bmatrix} 700 & -700 \\ -700 & 700 \end{bmatrix} \qquad \{R^{(2)}\} = \begin{Bmatrix} F_1^{(2)} \\ F_2^{(2)} \end{Bmatrix} \tag{E5}$$

Step 2. The global and local orientations of the elements are the same, so no transformation of the element stiffness matrices and the load vectors is needed.

Step 3. There are three nodes at the global level. The global stiffness matrix and load vector before incorporation of boundary conditions and loads can be written as in Equation (E6).

$$[K_G] = \begin{bmatrix} 3800 & -3800 & 0 \\ -3800 & 4500 & -700 \\ 0 & -700 & 700 \end{bmatrix} \qquad \{R_G\} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} + F_1^{(2)} \\ F_2^{(2)} \end{Bmatrix} \tag{E6}$$

Step 4. We note that there is no concentrated force at node B, as recognized in Equation (E7).

$$F_2^{(1)} + F_1^{(2)} = 0 \tag{E7}$$

Step 5. The force at both the element and global levels is in the direction of displacement at point C. Hence we can write Equation (E8).

$$F_2^{(2)} = P = 10 \text{ kips} \tag{E8}$$

Step 6. The displacement at A is zero, which corresponds to the first degree of freedom. We eliminate the first row and column to obtain the algebraic equations in the matrix form shown Equation (E9).

$$\begin{bmatrix} 4500 & -700 \\ -700 & 700 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} \tag{E9}$$

Step 7. Equation (E9) can be solved to obtain the results shown in Equation (E10).

$$u_2 = 2.6316(10^{-3}) \text{ in} \quad \text{and} \quad u_3 = 16.9173(10^{-3}) \text{ in} \tag{E10}$$

The displacement at C is the displacement of node 3.

ANS. $u_C = 0.00169$ in

Step 8. The stress at point B in each element can be found by using Equation (9.8a) as shown in Equation (E11) and (E12).

$$\sigma_B^{(1)} = E\left[u_1^{(1)}\frac{d\mathcal{L}_1}{dx} + u_2^{(1)}\frac{d\mathcal{L}_2}{dx}\right]\Bigg|_{x=10} = E\left[u_1\left(\frac{1}{-L_1}\right) + u_2\left(\frac{1}{L_1}\right)\right] = 10{,}000\left(\frac{2.6316(10^{-3})}{10}\right) \tag{E11}$$

The axial stress at point B in element 1 is

ANS. $\sigma_B^{(1)} = 2.63 \text{ ksi (T)}$

$$\sigma_B^{(2)} = E\left[u_1^{(2)}\frac{d\mathcal{L}_1}{dx} + u_2^{(2)}\frac{d\mathcal{L}_2}{dx}\right]\Bigg|_{x=10} = E\left[u_2\left(\frac{1}{-L_2}\right) + u_3\left(\frac{1}{L_2}\right)\right] = 10{,}000\left(\frac{14.2857(10^{-3})}{40}\right) \tag{E12}$$

The axial stress at point B in element 2 is

ANS. $\sigma_B^{(2)} = 3.57 \text{ ksi (T)}$

The strain energy in each element can be calculated as described in Section 9.3.6. The internal axial force in each element can be found from Equation (9.7c), as shown in Equation (E13) and (E14).

$$N^{(1)} = A^{(1)}\sigma_B^{(1)} = (3.8)(2.6316) = 10.0 \tag{E13}$$

$$N^{(2)} = A^{(2)}\sigma_B^{(2)} = (2.8)(3.5714) = 10.0 \tag{E14}$$

The strain energy in each of the elements can be found from Equation (9.19a) as shown in Equation (E15) and (E16).

$$U_A^{(1)} = \frac{1}{2}\left[u_2^{(1)} - u_1^{(1)}\right]N^{(1)} = \frac{1}{2}u_2 N^{(1)} = 0.01316 \text{ in}\cdot\text{kips} \tag{E15}$$

$$U_A^{(2)} = \frac{1}{2}\left[u_2^{(2)} - u_1^{(2)}\right]N^{(2)} = \frac{1}{2}[u_3 - u_2]N^{(2)} = 0.07143 \text{ in}\cdot\text{kips} \tag{E16}$$

Model 2

Step 1. From Equation (E1), the cross-sectional areas at the midpoints for the three elements shown in Figure 9.9*b* can be written as in Equation (E17).

$$A_1 = A(5) = 3.8 \text{ in}^2 \qquad A_2 = A(20) = 3.2 \text{ in}^2 \qquad A_3 = A(40) = 2.4 \text{ in}^2 \tag{E17}$$

Noting that $L_1 = 10$, $L_2 = 20$, and $L_3 = 20$, we obtain Equation (E18).

$$E_1A_1/L_1 = 3800 \qquad E_2A_2/L_2 = 1600 \qquad E_3A_3/L_3 = 1200 \tag{E18}$$

The element stiffness matrix and element load vector for element 1 are same as in Equation (E4). The element stiffness matrix and element load vector for the remaining two elements can be written by using Equation (9.21a), as shown in Equation (E19) and (E20).

$$[K^{(2)}] = \begin{bmatrix} 1600 & -1600 \\ -1600 & 1600 \end{bmatrix} \qquad \{R^{(2)}\} = \begin{Bmatrix} F_1^{(2)} \\ F_2^{(2)} \end{Bmatrix} \tag{E19}$$

$$[K^{(3)}] = \begin{bmatrix} 1200 & -1200 \\ -1200 & 1200 \end{bmatrix} \qquad \{R^{(3)}\} = \begin{Bmatrix} F_1^{(3)} \\ F_2^{(3)} \end{Bmatrix} \tag{E20}$$

Step 2. The global and local orientations of the elements are same, so no transformation of the element stiffness matrices and load vectors is needed.

Step 3. There are four nodes at the global level. The global stiffness matrix and load vector can be written as in Equation (E21).

$$[K_G] = \begin{bmatrix} 3800 & -3800 & 0 & 0 \\ -3800 & 5400 & -1600 & 0 \\ 0 & -1600 & 2800 & -1200 \\ 0 & 0 & -1200 & 1200 \end{bmatrix} \qquad \{R_G\} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} + F_1^{(2)} \\ F_2^{(2)} + F_1^{(3)} \\ F_2^{(3)} \end{Bmatrix} \tag{E21}$$

Step 4. We note that there is no concentrated force at nodes B and D in Figure 9.9(b). Hence by force equivalence, we obtain Equation (E22) and (E23).

$$F_2^{(1)} + F_1^{(2)} = 0 \tag{E22}$$

$$F_2^{(2)} + F_1^{(3)} = 0 \tag{E23}$$

The force at both the element and global levels is in the direction of displacement at point C. Hence we obtain Equation (E24).

$$F_2^{(3)} = P = 10 \text{ kips} \tag{E24}$$

Step 5. The displacement at A is zero, which corresponds to the first degree of freedom. We eliminate the first row and column to obtain the algebraic equations in matrix form, as follow:

$$\begin{bmatrix} 5400 & -1600 & 0 \\ -1600 & 2800 & -1200 \\ 0 & -1200 & 1200 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 10 \end{Bmatrix} \tag{E25}$$

Step 6. The displacements in Equation (E25) are solved in Equation (E26).

$$u_2 = 2.6316(10^{-3}) \text{ in} \qquad u_3 = 8.8816(10^{-3}) \text{ in} \qquad u_4 = 17.2149(10^{-3}) \text{ in} \tag{E26}$$

The displacement at point C is the displacement of node 4.

ANS. $u_C = 0.00172$ in

The stress at point B in element 1 is as given by Equation (E11) because the displacement at node 2 did not change in this model. The stress in element 2 can be found by using Equation (9.7b) as shown in Equation (E27).

$$\sigma_B^{(2)} = E\left[u_1^{(2)}\frac{d\mathcal{L}_1}{dx} + u_2^{(2)}\frac{d\mathcal{L}_2}{dx}\right]\Bigg|_{x=10} = E\left[u_2\left(\frac{1}{-L_2}\right) + u_3\left(\frac{1}{L_2}\right)\right] = 10{,}000\left(\frac{6.25(10^{-3})}{20}\right) \tag{E27}$$

The axial stress at point B in element 2 is

ANS. $\sigma_B^{(2)} = 3.13$ ksi (T)

The strain energy in element 1 is as given by Equation (E15) because the node 2 displacement of this model is the same as for model 1. The internal forces in the remaining two elements are as shown in Equation (E28) and (E29).

$$N^{(2)} = A^{(2)}\sigma_B^{(2)} = 10.0 \tag{E28}$$

$$N^{(3)} = A^{(3)}\sigma_B^{(3)} = 10.0 \tag{E29}$$

The strain energy in each element can be found from Equation (9.19a), as shown in Equation (E30) and (E31).

$$U_A^{(2)} = \frac{1}{2}[u_2^{(2)} - u_1^{(2)}]N^{(2)} = \frac{1}{2}[u_3 - u_2]N^{(2)} = 0.03125 \text{ in} \cdot \text{kip} \tag{E30}$$

$$U_A^{(3)} = \frac{1}{2}[u_2^{(3)} - u_1^{(3)}]N^{(3)} = \frac{1}{2}[u_4 - u_3]N^{(3)} = 0.041667 \text{ in} \cdot \text{kip} \tag{E31}$$

Step 7. The analytical results for displacement and stress for the tapered axial rod shown in Figure 9.8 are given in Equation (E32).

$$\sigma_{xx}(x) = \frac{2.5}{(1 - 0.01x)} \text{ ksi} \qquad \text{and} \qquad u(x) = -0.025\ ln(1 - 0.01x) \text{ in} \tag{E32}$$

The strain energy of the entire rod can be found by using the fact that the internal force is $N = 10$ kN, as shown in Equation (E33).

$$U_A = \frac{1}{2}\int_0^{50}\left(EA\frac{du}{dx}(x)\right)\frac{du}{dx}(x)dx = \frac{1}{2}\int_0^{50} N(x)\frac{du}{dx}(x)dx = \frac{1}{2}\int_0^{50}(10)\frac{du}{dx}(x)dx \quad \text{or}$$

$$U_A = 5[u(50) - u(0)] = -0.086643 \text{ in} \cdot \text{kip} \tag{E33}$$

The potential energy is twice the negative value of the strain energy at equilibrium, as shown in Equation (7.36). The analytical results and the results using the two FEM models are shown in Table 9.1.

	u_C (inches) (10^{-3})	Stress at B: ρ_B (ksi)			Strain Energy: U_A			Total Potential energy Ω
		Left of B	Right of B	Average	Element 1	Element 2	Element 3	
Model 1	16.917	2.632	3.571	3.101	0.01316	0.07143		−0.16917
Model 2	17.215	2.632	3.125	2.878	0.01316	0.03125	0.04167	−0.17215
Analytical	17.329	2.778	2.778	2.778				−0.17329

COMMENTS

1. The total potential energy of model 2 is less than that of model 1, thus we expect the results of model 2 to be better than that of model 1 and this expectation is validated by the results in every category of Table 9.1. However, commercial codes do not calculate the total potential energy of a structure. Also the total energy cannot tell us if results in a particular area of interest will improve or not. There are several other indicators that can be used to make decisions to refine a mesh or not in a given area

2. For model 1 the strain energy of element 2 is significantly larger than element 1. The more evenly the strain energy is distributed over the elements in a mesh the better will be the results. A uniform value of strain energy over the elements will produce the best results for a given number of degrees of freedom. Thus, regions containing elements with high-strain energy should be refined.
3. There is a stress discontinuity at point B in both models. This should not be the case as B is not an interface of two materials nor is there a concentrated force to cause the stress to jump. Thus, the discontinuity is an artifact of the finite element model. The discontinuity in model 2 however is smaller than the discontinuity in model 1. This is another indicator that the regions (elements) showing large stress discontinuity must be refined.
4. The average stress value at B in both models is closer to the analytical solution than stress value in either elements. Thus, for purpose of using FEM results for design and analysis the average values should be used.
5. The reaction force at A can be found noting that $F_1^{(1)} = R_A$ in the first row of Equation (E6) as shown in Equation (E34).

$$R_A = u_1^{(1)}(3800) + u_2^{(1)}(-3800) = u_1(3800) + u_2(-3800) = \;= 2.6316(10^{-3})(-3800) = -10 \text{ kN} \tag{E34}$$

The minus sign indicates that the force R_A acts opposite to the positive direction $u_1^{(1)}$. In other words, the reaction is to the left, which we can see intuitively is the correct direction. In interpreting results from FEM models, one must note carefully the local and global coordinate directions

We record the following observations

- A good FEM mesh should not show large differences in element strain energy.
- A good FEM mesh should not show large discontinuities in stress values across element boundaries unless the element boundary is an interface of two materials or a concentrated force is applied at the element end.
- Average values of stresses at the element boundary should be used for purposes of design and analysis.
- Care must be taken in interpreting FEM results: note the orientation of the local and global coordinates and whether the variable reported is in the local or global coordinates.

EXAMPLE 9.2

A force of $F = 20$ kN is applied to a roller that slides inside a slot as shown in Figure 9.10. Both bars have a cross-sectional area of $A = 100$ mm^2 and a modulus of elasticity $E = 200$ GPa. Bars AP and BP have lengths of $L_{AP} = 200$ mm and $L_{BP} = 250$ mm, respectively. Determine the displacement of the roller and the reaction force on the roller, using linear elements to represent each bar.

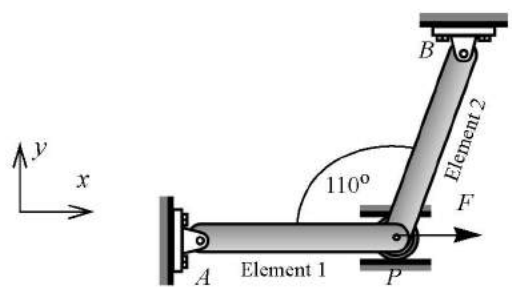

Figure 9.10 Two-bar structure for Example 9.2.

PLAN

We will model each pin as able to displace in the x as well as the y direction to account for the fact that element 2 is at an angle to the x axis. We will follow the steps outlined in Section 9.3.9 to obtain the results.

SOLUTION

Step 1. Equation (E1) and (E2) can be written from the given information.

$$\frac{E_1 A_1}{L_1} = \frac{(200)(10^9)(100)(10^{-6})}{(200)(10^{-3})} = (100)(10^6) \tag{E1}$$

$$\frac{E_2 A_2}{L_2} = \frac{(200)(10^9)(100)(10^{-6})}{(250)(10^{-3})} = (80)(10^6) \tag{E2}$$

We can use Equation, (9.21a) to write the element stiffness matrix and load vector in the local coordinates for the two elements, as shown in Equation (E3) and (E4).

$$[K^{(1)}] = \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix}(10^6) \qquad \{R^{(1)}\} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \end{Bmatrix} \tag{E3}$$

$$[K^{(2)}] = \begin{bmatrix} 80 & -80 \\ -80 & 80 \end{bmatrix}(10^6) \qquad \{R^{(2)}\} = \begin{Bmatrix} F_1^{(2)} \\ F_2^{(2)} \end{Bmatrix} \tag{E4}$$

Step 2. Element 1 is in the x direction, but in the global coordinate each node has 2 degrees of freedom (u and v), and thus the element stiffness matrix and load vector in the global coordinate system can be written as in Equation (E5).

$$[K_G^{(1)}] = \begin{bmatrix} 100 & 0 & -100 & 0 \\ 0 & 0 & 0 & 0 \\ -100 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}(10^6) \qquad \{R^{(1)}\} = \begin{Bmatrix} F_1^{(1)} \\ 0 \\ F_2^{(1)} \\ 0 \end{Bmatrix} \tag{E5}$$

We note that the element 2 makes an angle of 70° with the x axis. The transformation matrix can be written as in Equation (E6).

$$[T] = \begin{bmatrix} \cos 70 & \sin 70 & 0 & 0 \\ 0 & 0 & \cos 70 & \sin 70 \end{bmatrix} = \begin{bmatrix} 0.3420 & 0.9397 & 0 & 0 \\ 0 & 0 & 0.3420 & 0.9397 \end{bmatrix} \tag{E6}$$

The element stiffness matrix for element 2 in the global coordinate system can be calculated as shown in Equation (E7).

$$[K_G^{(2)}] = [T]^T[K^{(2)}][T] = [T]^T\begin{bmatrix} 80 & -80 \\ -80 & 80 \end{bmatrix}\begin{bmatrix} 0.3420 & 0.9397 & 0 & 0 \\ 0 & 0 & 0.3420 & 0.9397 \end{bmatrix}(10^6) \text{ or}$$

$$[K_G^{(2)}] = \begin{bmatrix} 0.3420 & 0 \\ 0.9397 & 0 \\ 0 & 0.3420 \\ 0 & 0.9397 \end{bmatrix}\begin{bmatrix} 27.362 & 75.175 & -27.362 & -75.175 \\ -27.362 & -75.175 & 27.362 & 75.175 \end{bmatrix}(10^6)$$

$$[K_G^{(2)}] = \begin{bmatrix} 9.3582 & 25.711 & -9.3582 & 25.711 \\ 25.711 & 70.642 & -25.711 & -70.642 \\ -9.3582 & -25.711 & 9.3582 & 25.711 \\ 25.711 & -70.642 & 25.711 & 70.642 \end{bmatrix}(10^6) \tag{E7}$$

The element load vector for element 2 can be calculated as shown in Equation (E8).

$$\{R_G^{(2)}\} = [T]^T\{R^{(2)}\} = \begin{bmatrix} 0.3420 & 0 \\ 0.9397 & 0 \\ 0 & 0.3420 \\ 0 & 0.9397 \end{bmatrix}\begin{Bmatrix} F_1^{(2)} \\ F_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0.3420F_1^{(2)} \\ 0.9397F_1^{(2)} \\ 0.3420F_2^{(2)} \\ 0.9397F_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} F_{1x}^{(2)} \\ F_{1y}^{(2)} \\ F_{2x}^{(2)} \\ F_{2y}^{(2)} \end{Bmatrix} \tag{E8}$$

Step 3. With three nodes there are 6 degrees of freedom. The potential energy for each element can be written as shown in Equations (E9) and (E10).

$$\delta\Omega^{(1)} = \{\delta u_G^{(1)}\}^T\left(\begin{bmatrix} 100 & 0 & -100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -100 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}(10^6)\{u_G^{(1)}\} - \begin{Bmatrix} F_1^{(1)} \\ 0 \\ F_2^{(1)} \\ 0 \\ 0 \\ 0 \end{Bmatrix}\right) \tag{E9}$$

$$\delta\Omega^{(2)} = \{\delta u_G^{(2)}\}^T\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9.3582 & 25.711 & -9.3582 & 25.711 \\ 0 & 0 & 25.711 & 70.642 & -25.711 & -70.642 \\ 0 & 0 & -9.3582 & -25.711 & 9.3582 & 25.711 \\ 0 & 0 & 25.711 & -70.642 & 25.711 & 70.642 \end{bmatrix}(10^6)\{u_G^{(2)}\} - \begin{Bmatrix} 0 \\ 0 \\ F_{1x}^{(2)} \\ F_{1y}^{(2)} \\ F_{2x}^{(2)} \\ F_{2y}^{(2)} \end{Bmatrix}\right) \tag{E10}$$

The total potential energy of the system can be obtained by adding the potential energies in Equations (E9) and (E10) to obtain Equation (E11).

$$\delta\Omega = \begin{Bmatrix} \delta u_1 \\ \delta v_1 \\ \delta u_2 \\ \delta v_2 \\ \delta u_3 \\ \delta v_3 \end{Bmatrix}^T \left(\begin{bmatrix} 100 & 0 & -100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -100 & 0 & 109.3582 & 25.711 & -9.3582 & 25.711 \\ 0 & 0 & 25.711 & 70.642 & -25.711 & -70.642 \\ 0 & 0 & -9.3582 & -25.711 & 9.3582 & 25.711 \\ 0 & 0 & 25.711 & -70.642 & 25.711 & 70.642 \end{bmatrix} (10^6) \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} - \begin{Bmatrix} F_1^{(1)} \\ 0 \\ F_2^{(1)} + F_{1x}^{(2)} \\ F_{1y}^{(2)} \\ F_{2x}^{(2)} \\ F_{2y}^{(2)} \end{Bmatrix} \right) \tag{E11}$$

Step 4. Noting that the applied load at the roller is in the x direction and that there are reaction forces at points A and B, we obtain the load equivalences shown in Equation (E12),

$$F_1^{(1)} = R_A \qquad F_2^{(1)} + F_{1x}^{(2)} = 20(10^3)\text{ N} \qquad F_{1y}^{(2)} = R_P \qquad F_{2x}^{(2)} = R_{Bx} \qquad F_{2y}^{(2)} = R_{By} \tag{E12}$$

where R_A and R_P are the reaction forces at A and P, and R_{Bx} and R_{By} are the reaction forces at B in the x and y directions. Substituting Equation (E12) into Equation (E11), we obtain Equation (E13).

$$\delta\Omega = \begin{Bmatrix} \delta u_1 \\ \delta v_1 \\ \delta u_2 \\ \delta v_2 \\ \delta u_3 \\ \delta v_3 \end{Bmatrix}^T \left(\begin{bmatrix} 100 & 0 & -100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -100 & 0 & 109.3582 & 25.711 & -9.3582 & 25.711 \\ 0 & 0 & 25.711 & 70.642 & -25.711 & -70.642 \\ 0 & 0 & -9.3582 & -25.711 & 9.3582 & 25.711 \\ 0 & 0 & 25.711 & -70.642 & 25.711 & 70.642 \end{bmatrix} (10^6) \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} - \begin{Bmatrix} R_A \\ 0 \\ 20(10^3) \\ R_P \\ R_{Bx} \\ R_{By} \end{Bmatrix} \right) \tag{E13}$$

Step 5. The boundary conditions on displacements are given in Equation (E14).

$$\begin{matrix} u_1 = 0 & v_1 = 0 & v_2 = 0 & u_3 = 0 & v_3 = 0 \\ \delta u_1 = 0 & \delta v_1 = 0 & \delta v_2 = 0 & \delta u_3 = 0 & \delta v_3 = 0 \end{matrix} \tag{E14}$$

Substituting Equation (E14) into Equation (E13) and equating $\delta\Omega = 0$, we obtain Equation (E15).

$$109.3582(10^6)u_2 = 20(10^3) \qquad \text{or} \qquad u_2 = 0.1829(10^{-3})\text{ m} \tag{E15}$$

The displacement of the roller is

ANS. $u_B = 0.1829$ mm

The reaction force can be calculated from the fourth row of Equation (E13) as shown in Equation (E16).

$$25.711(10^6)u_2 = R_P \qquad \text{or} \qquad R_P = [25.711(10^6)][0.1829(10^{-3})] = 4.702(10^3)\text{ N} \tag{E16}$$

ANS. $R_P = 4.7$ kN

COMMENT

If the roller were not in the slot, it would be free to displace in the y direction, and v_2 would not be zero. In such a case, Equation (E13) would yield two equations in the two unknowns u_2 and v_2, which could be solved for the displacement of the roller. See Problem 9.7.

9.4 CIRCULAR SHAFT ELEMENTS

It was seen in the Raleigh-Ritz method that the derivation of a stiffness matrix and load vectors for torsion of circular shafts is similar to that for axial members. The stiffness matrix and load vectors can be obtained by replacing (a) the axial rigidity EA by torsional rigidity GJ, (b) the distributed axial force $p_x(x)$ by the distributed torque $t(x)$, and (c) the concentrated axial forces F_i by the concentrated torques T_i. The assembly and solution procedure is the same as for axial members.

9.5 SYMMETRIC BEAM ELEMENTS

The deflection v and its derivative dv/dx must be continuous at all points on the beam, including the element ends. Lagrange polynomials cannot be used for representing v because as elaborated in Section 9.2, this would result in a slope that was discontinuous at the element end irrespective of the order of polynomial used. To overcome this problem of continuity of slope, we must define v and dv/dx as degrees of freedom at the element ends. With 2 degrees of freedom at each end, we have a total of 4 degrees of freedom in the element. A cubic polynomial has four unknown constants that can be solved in terms of the 4 degrees of freedom. We write v(x) and the four conditions as shown in Equation (9.22) and (9.23).

$$v(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 \tag{9.22}$$

$$v(x_1) = v_1^{(e)} \qquad \frac{dv}{dx}(x_1) = \theta_1^{(e)} \qquad v(x_2) = v_2^{(e)} \qquad \frac{dv}{dx}(x_2) = \theta_2^{(e)} \tag{9.23}$$

Substituting Equation (9.22) into the four conditions in Equation (9.23), solving the constants C's, and substituting the result back into Equation (9.22), we obtain Equation (9.24a) and (9.24c).

$$v(x) = f_1(x)v_1^{(e)} + f_2(x)\theta_1^{(e)} + f_3(x)v_2^{(e)} + f_4(x)\theta_2^{(e)} \tag{9.24a}$$

$$f_1(x) = 1 - 3\left(\frac{x-x_1}{L}\right)^2 + 2\left(\frac{x-x_1}{L}\right)^3 \qquad f_2(x) = L\left[\left(\frac{x-x_1}{L}\right) - 2\left(\frac{x-x_1}{L}\right)^2 + \left(\frac{x-x_1}{L}\right)^3\right] \tag{9.24b}$$

$$f_3(x) = 3\left(\frac{x-x_1}{L}\right)^2 - 2\left(\frac{x-x_1}{L}\right)^3 \qquad f_4(x) = L\left[-\left(\frac{x-x_1}{L}\right)^2 + \left(\frac{x-x_1}{L}\right)^3\right] \tag{9.24c}$$

The above interpolation functions belong to a class called *Hermite polynomials*. Figure 9.11 shows plots of Hermite polynomials. Note the zero value of the function and the zero value of the slopes. These values are a consequence of the fact that Equation (9.24a) must satisfy the conditions in Equation (9.23)

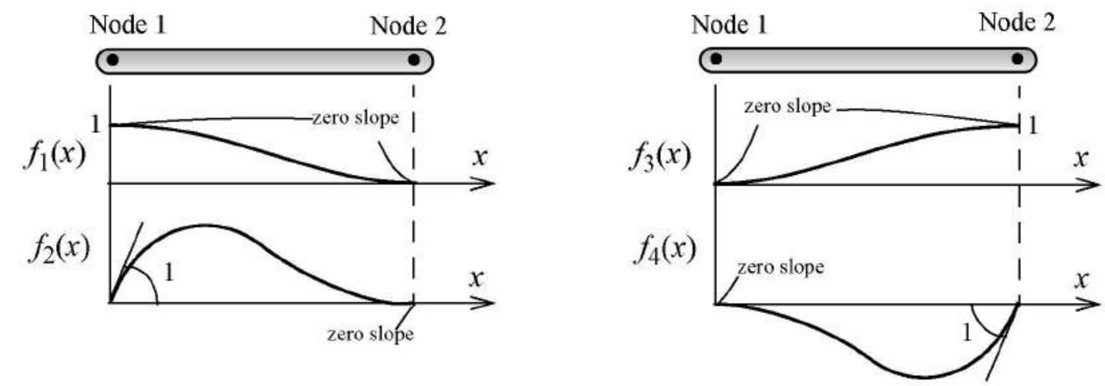

Figure 9.11 Hermite polynomials.

The stiffness matrix was given by Equation (7.38c). To obtain the element load vector, we assume that point forces and moments can be applied only at element end nodes. This requirement is easily met during mesh creation, where we create an element such that the point forces and moments are at the end. From this requirement and from Equation (7.38d), we obtain the right-hand-side vector as shown in Equation (9.25a).

$$R_j^{(e)} = \int_0^L p_y(x) f_j(x)\,dx + F_1^{(e)} f_j(x_1) + F_2^{(e)} f_j(x_2) + M_1^{(e)} \frac{df_j}{dx}(x_1) + M_2^{(e)} \frac{df_j}{dx}(x_2) \tag{9.25a}$$

From Figure 9.11 we have that $f_j(x_1)$ is zero except when $j = 1$, and $f_j(x_2)$ is zero except when $j = 2$. Similarly df_j/dx at x_1 is zero except when $j = 2$, and df_j/dx at x_2 is zero except when $j = 4$.

Figure 9.12 shows the local coordinate system for the beam element and the positive directions for deflection and slopes. For the work potential to be positive, the nodal forces and moments must be positive in the same direction as the deflection and slope..

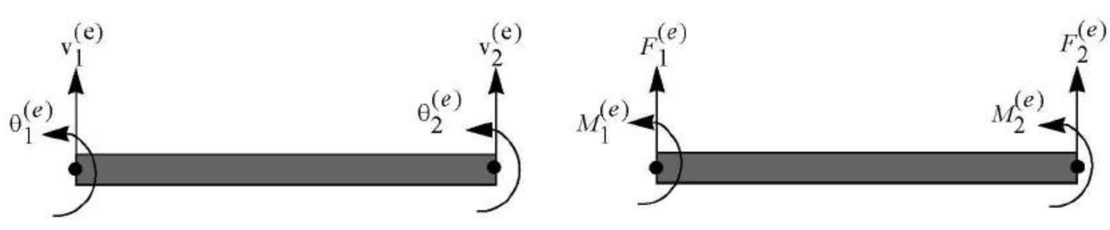

Figure 9.12 Positive directions on a beam element for (a) deflection and slope and (b) forces and moments.

In writing the element stiffness matrix and element load vectors, we will assume the following to account for the possibility of varying distributed loads, cross-sectional areas, and moduli of elasticity.

- The distributed load p_y is evaluated at the midpoint of the element and has a uniform value p_0.
- The area moment of inertia I is evaluated for the cross section at the midpoint of the element.
- The modulus of elasticity E is constant over the element.

SubstitutingEquations (9.24b) and (9.24c) into Equations (7.38c)) and (9.25a), we obtain the element stiffness matrix and load vectors shown in Equation (9.26a) and (9.26b). We also show the displacement vector with 4 degrees of freedom.

$$\{v^{(e)}\} = \begin{Bmatrix} v_1^{(e)} \\ \theta_1^{(e)} \\ v_2^{(e)} \\ \theta_2^{(e)} \end{Bmatrix} \qquad [K^{(e)}] = \frac{2E^{(e)}I^{(e)}}{(L^{(e)})^3}\begin{bmatrix} 6 & 3L^{(e)} & -6 & 3L^{(e)} \\ 3L^{(e)} & 2(L^{(e)})^2 & -3L^{(e)} & (L^{(e)})^2 \\ -6 & -3L^{(e)} & 6 & -3L^{(e)} \\ 3L^{(e)} & (L^{(e)})^2 & -3L^{(e)} & 2L^{(e)2} \end{bmatrix} \tag{9.26a}$$

$$\{R^{(e)}\} = \frac{p_0^{(e)}L^{(e)}}{12}\begin{Bmatrix} 6 \\ L^{(e)} \\ 6 \\ -L^{(e)} \end{Bmatrix} + \begin{Bmatrix} F_1^{(e)} \\ M_1^{(e)} \\ F_2^{(e)} \\ M_2^{(e)} \end{Bmatrix} \tag{9.26b}$$

The virtual variation of potential energy for the element can be written in compact matrix form as shown in Equation (9.26c).

$$\delta\Omega_B^{(e)} = \{\delta v^{(e)}\}^T([K^{(e)}]\{v^{(e)}\} - \{R^{(e)}\}) \tag{9.26c}$$

The assembly and solution process, which is as described Section 9.3 for axial members, is elaborated in Example 9.3.

EXAMPLE 9.3

For the beam and loading shown in Figure 9.13, use two equal elements to determine the deflection and slope at B and the reaction force and moment at A. Assume that EI is constant for the beam.

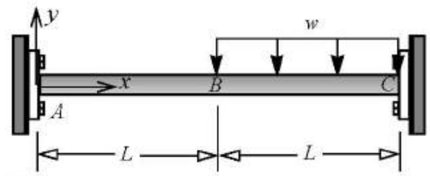

Figure 9.13 Beam and loading for Example 9.3.

PLAN

There are three nodes on the beam, resulting in a total of 6 degrees of freedom before boundary conditions are imposed. We follow the procedure outlined in Section 9.3.9.

SOLUTION

Step 1. Using Equation (9.26a) and noting that $p_0 = -w$, we write the load vectors for the two elements as shown in Equation (E1).

$$\{R^{(1)}\} = \begin{Bmatrix} F_1^{(1)} \\ M_1^{(1)} \\ F_2^{(1)} \\ M_2^{(1)} \end{Bmatrix} \quad \text{and} \quad \{R^{(2)}\} = -\left(\frac{wL}{12}\right)\begin{Bmatrix} 6 \\ L \\ 6 \\ -L \end{Bmatrix} + \begin{Bmatrix} F_1^{(2)} \\ M_1^{(2)} \\ F_2^{(2)} \\ M_2^{(2)} \end{Bmatrix} \tag{E1}$$

The element stiffness matrix is the same for both elements. From Equation (9.26a), the element stiffness matrix can be written as shown in Equation (E2).

$$[K^{(1)}] = [K^{(2)}] = \frac{2EI}{L^3}\begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \tag{E2}$$

Step 2. We note that the relationships between the local displacements and slopes and the global displacements and slopes are as given by Equation (E3).

$$\begin{matrix} v_1^{(1)} = v_A & v_2^{(1)} = v_B & v_1^{(2)} = v_B & v_2^{(2)} = v_C \\ \theta_1^{(1)} = \theta_A & \theta_2^{(1)} = \theta_B & \theta_1^{(2)} = \theta_B & \theta_2^{(2)} = \theta_C \end{matrix} \tag{E3}$$

Step 3. The variation of potential energy function for the two elements can be written as shown in Equation (E4) and (E5).

$$\delta\Omega^{(1)} = \begin{Bmatrix} \delta v_1^{(1)} \\ \delta\theta_1^{(1)} \\ \delta v_2^{(1)} \\ \delta\theta_2^{(1)} \end{Bmatrix}^T \left(\frac{2EI}{L^3} \begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \begin{Bmatrix} v_1^{(1)} \\ \theta_1^{(1)} \\ v_2^{(1)} \\ \theta_2^{(1)} \end{Bmatrix} - \begin{Bmatrix} F_1^{(1)} \\ M_1^{(1)} \\ F_2^{(1)} \\ M_2^{(1)} \end{Bmatrix} \right)$$

$$\delta\Omega^{(1)} = \begin{Bmatrix} \delta v_A \\ \delta\theta_A \\ \delta v_B \\ \delta\theta_B \\ \delta v_C \\ \delta\theta_C \end{Bmatrix}^T \left(\frac{2EI}{L^3} \begin{bmatrix} 6 & 3L & -6 & 3L & 0 & 0 \\ 3L & 2L^2 & -3L & L^2 & 0 & 0 \\ -6 & -3L & 6 & -3L & 0 & 0 \\ 3L & L^2 & -3L & 2L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_A \\ \theta_A \\ v_B \\ \theta_B \\ v_C \\ \theta_C \end{Bmatrix} - \begin{Bmatrix} F_1^{(1)} \\ M_1^{(1)} \\ F_2^{(1)} \\ M_2^{(1)} \\ 0 \\ 0 \end{Bmatrix} \right) \tag{E4}$$

$$\delta\Omega^{(2)} = \begin{Bmatrix} \delta v_1^{(2)} \\ \delta\theta_1^{(2)} \\ \delta v_2^{(2)} \\ \delta\theta_2^{(2)} \end{Bmatrix}^T \left(\frac{2EI}{L^3} \begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \begin{Bmatrix} v_1^{(2)} \\ \theta_1^{(2)} \\ v_2^{(2)} \\ \theta_2^{(2)} \end{Bmatrix} - \begin{Bmatrix} F_1^{(2)} \\ M_1^{(2)} \\ F_2^{(2)} \\ M_2^{(2)} \end{Bmatrix} + \frac{wL}{12} \begin{Bmatrix} 6 \\ L \\ 6 \\ -L \end{Bmatrix} \right)$$

$$\delta\Omega^{(2)} = \begin{Bmatrix} \delta v_A \\ \delta\theta_A \\ \delta v_B \\ \delta\theta_B \\ \delta v_C \\ \delta\theta_C \end{Bmatrix}^T \left(\frac{2EI}{L^3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 3L & -6 & 3L \\ 0 & 0 & 3L & 2L^2 & -3L & L^2 \\ 0 & 0 & -6 & -3L & 6 & -3L \\ 0 & 0 & 3L & L^2 & -3L & 2L^2 \end{bmatrix} \begin{Bmatrix} v_A \\ \theta_A \\ v_B \\ \theta_B \\ v_C \\ \theta_C \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ F_1^{(2)} \\ M_1^{(2)} \\ F_2^{(2)} \\ M_2^{(2)} \end{Bmatrix} + \frac{wL}{12} \begin{Bmatrix} 0 \\ 0 \\ 6 \\ L \\ 6 \\ -L \end{Bmatrix} \right) \tag{E5}$$

The total potential energy of the beam, $\delta\Omega = \delta\Omega^{(1)} + \delta\Omega^{(2)}$, can be written as in Equation (E6).

$$\delta\Omega = \begin{Bmatrix} \delta v_A \\ \delta\theta_A \\ \delta v_B \\ \delta\theta_B \\ \delta v_C \\ \delta\theta_C \end{Bmatrix}^T \left(\frac{2EI}{L^3} \begin{bmatrix} 6 & 3L & -6 & 3L & 0 & 0 \\ 3L & 2L^2 & -3L & L^2 & 0 & 0 \\ -6 & -3L & 12 & 0 & -6 & 3L \\ 3L & L^2 & 0 & 4L^2 & -3L & L^2 \\ 0 & 0 & -6 & -3L & 6 & -3L \\ 0 & 0 & 3L & L^2 & -3L & 2L^2 \end{bmatrix} \begin{Bmatrix} v_A \\ \theta_A \\ v_B \\ \theta_B \\ v_C \\ \theta_C \end{Bmatrix} - \begin{Bmatrix} F_1^{(1)} \\ M_1^{(1)} \\ F_2^{(1)} + F_1^{(2)} \\ M_2^{(1)} + M_1^{(2)} \\ F_2^{(2)} \\ M_2^{(2)} \end{Bmatrix} + \frac{wL}{12} \begin{Bmatrix} 0 \\ 0 \\ 6 \\ L \\ 6 \\ -L \end{Bmatrix} \right) \tag{E6}$$

Step 4. We note that there are no external forces or moments at B and that the relationships between the element nodal forces and moments and the reaction forces and moments at A and C can be written as in Equation (E7).

$$\begin{aligned} F_1^{(1)} &= R_A & F_2^{(1)} + F_1^{(2)} &= 0 & F_2^{(2)} &= R_C \\ M_1^{(1)} &= M_A & M_2^{(1)} + M_1^{(2)} &= 0 & M_2^{(2)} &= M_C \end{aligned} \tag{E7}$$

Substituting Equation (E7) into Equation (E6), we obtain the potential energy function as shown in Equation (E8).

$$\delta\Omega = \begin{Bmatrix} \delta v_A \\ \delta\theta_A \\ \delta v_B \\ \delta\theta_B \\ \delta v_C \\ \delta\theta_C \end{Bmatrix}^T \left(\frac{2EI}{L^3}\begin{bmatrix} 6 & 3L & -6 & 3L & 0 & 0 \\ 3L & 2L^2 & -3L & L^2 & 0 & 0 \\ -6 & -3L & 12 & 0 & -6 & 3L \\ 3L & L^2 & 0 & 4L^2 & -3L & L^2 \\ 0 & 0 & -6 & -3L & 6 & -3L \\ 0 & 0 & 3L & L^2 & -3L & 2L^2 \end{bmatrix} \begin{Bmatrix} v_A \\ \theta_A \\ v_B \\ \theta_B \\ v_C \\ \theta_C \end{Bmatrix} - \begin{Bmatrix} R_A \\ M_A \\ 0 \\ 0 \\ R_C \\ M_C \end{Bmatrix} + \frac{wL}{12}\begin{Bmatrix} 0 \\ 0 \\ 6 \\ L \\ 6 \\ -L \end{Bmatrix} \right) \tag{E8}$$

The displacement and slopes at A and C are zero as shown in Equation (E9).

$$\begin{matrix} v_A = 0 & \theta_A = 0 & v_C = 0 & \theta_C = 0 \\ \delta v_A = 0 & \delta\theta_A = 0 & \delta v_C = 0 & \delta\theta_C = 0 \end{matrix} \tag{E9}$$

Substituting Equation (E9) into Equation (E8), we can write the potential energy function as Equation (E10).

$$\delta\Omega = \begin{Bmatrix} \delta v_B \\ \delta\theta_B \end{Bmatrix}^T \left(\frac{2EI}{L^3}\begin{bmatrix} 12 & 0 \\ 0 & 4L^2 \end{bmatrix}\begin{Bmatrix} v_B \\ \theta_B \end{Bmatrix} + \frac{wL}{12}\begin{Bmatrix} 6 \\ L \end{Bmatrix} \right) \tag{E10}$$

By the principle of minimum potential energy, the virtual variation of potential energy due to virtual displacement must be zero (i.e., $\delta\Omega = 0$). Since the virtual displacement cannot be zero, the remaining terms in the brackets in Equation (E10) must be zero, and we obtain the set of algebraic equations shown in Equation (E11).

$$\frac{2EI}{L^3}\begin{bmatrix} 12 & 0 \\ 0 & 4L^2 \end{bmatrix}\begin{Bmatrix} v_B \\ \theta_B \end{Bmatrix} = -\left(\frac{wL}{12}\right)\begin{Bmatrix} 6 \\ L \end{Bmatrix} \tag{E11}$$

Solving the above two equations shown in Equation (E11), we obtain the results shown in Equation (E12).

$$\textbf{ANS.} \quad v_B = -\left(\frac{wL^4}{48EI}\right) \qquad \theta_B = -\left(\frac{wL^3}{96EI}\right) \tag{E12}$$

The reaction force and moments at A can be found from the first two rows of Equation (E8) as shown in Equation (E13) and (E14).

$$R_A = \frac{2EI}{L^3}[-6v_B + 3L\theta_B] = \frac{2EI}{L^3}\left[6\left(\frac{wL^4}{48EI}\right) - 3L\left(\frac{wL^3}{96EI}\right)\right] \tag{E13}$$

$$\textbf{ANS.}\ R_A = \frac{3}{16}wL$$

$$M_A = \frac{2EI}{L^3}[-3Lv_B + L^2\theta_B] = \frac{2EI}{L^3}\left[3L\left(\frac{wL^4}{48EI}\right) - L^2\left(\frac{wL^3}{96EI}\right)\right] \tag{E14}$$

$$\textbf{ANS.}\ M_A = \frac{5}{48}wL^2$$

COMMENTS

1. The primary difference in the solution procedure in this example and in the axial rod problem of is that two rows and two columns are added instead of one at the shared node of the two elements. This is not surprising, since the shared node has 2 degrees of freedom.
2. In accordance with Figure 9.12 the negative sign on deflection at B implies that it is downward, and a negative sign on slope implies clockwise rotation, which makes intuitive sense if we visualize the deformed shape.
3. In accordance with Figure 9.12, the positive signs for reactions implies that force is upward and the moment is counter-clockwise.

9.6 FINITE ELEMENT EQUATIONS IN TWO-DIMENSIONS

In this section the equations for the finite element method in two dimensions are presented. The strain energy density in two dimensions can be written in matrix form as shown in Equation (9.27a) and (9.27b).

$$U_0 = \frac{1}{2}[\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \tau_{xy}\gamma_{xy}] = \frac{1}{2}\{\sigma\}^T\{\varepsilon\} \tag{9.27a}$$

$$\{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{9.27b}$$

Hooke's law for plane stress [Equation (8.4a)] and plane strain [Equation (8.5a)] can be written in matrix form as shown in Equation (9.28a) through (9.28c).

$$\{\sigma\} = [\boldsymbol{E}]\{\varepsilon\} \tag{9.28a}$$

$$[\boldsymbol{E}] = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix} \quad \text{for plane strain} \tag{9.28b}$$

$$[\boldsymbol{E}] = \frac{E}{(1-2\nu)(1+\nu)}\begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \dfrac{(1-2\nu)}{2} \end{bmatrix} \quad \text{for plane strain} \tag{9.28c}$$

Substituting Equation (9.28a) into Equation (9.27a), we obtain Equation (9.28d).

$$U_0 = \frac{1}{2}\{\sigma\}^T\{\varepsilon\} = \frac{1}{2}\{\varepsilon\}^T[\boldsymbol{E}]^T\{\varepsilon\} = \frac{1}{2}\{\varepsilon\}^T[\boldsymbol{E}]\{\varepsilon\} \tag{9.28d}$$

The strain–displacement relationships, Equations (8.1a), (8.1b), and (8.1d), can be written in matrix form as shown in Equation (9.29a).

$$\{\varepsilon\} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}\begin{Bmatrix} u \\ \mathrm{v} \end{Bmatrix} \tag{9.29a}$$

The displacement at the element level can be approximated by using kinematically admissible functions as shown in Equation (9.29b).

$$u^{(e)}(x) = \sum_{i=1}^{n} u_i^{(e)} f_i(x, y) \qquad \text{and} \qquad \mathrm{v}^{(e)}(x) = \sum_{i=1}^{n} \mathrm{v}_i^{(e)} f_i(x, y) \tag{9.29b}$$

Equation (9.29b) can be written in matrix form as shown in Equation (9.29c) and (9.29d).

$$\begin{Bmatrix} u^{(e)} \\ \mathrm{v}^{(e)} \end{Bmatrix} = \begin{bmatrix} f_1 & 0 & f_2 & 0 & \dots & \dots & f_n & 0 \\ 0 & f_1 & 0 & f_2 & \dots & \dots & 0 & f_n \end{bmatrix}\{d^{(e)}\} \tag{9.29c}$$

$$\{d^{(e)}\}^T = \{u_1^{(e)}\mathrm{v}_1^{(e)},\ u_2^{(e)}\mathrm{v}_2^{(e)},\ \dots,\ u_n^{(e)}\mathrm{v}_n^{(e)}\} \tag{9.29d}$$

Substituting Equation (9.29c) and (9.29d) into Equation (9.29a), we can write the strain in an element as shown in Equation (9.30a) and (9.30b).

$$\{\varepsilon^{(e)}\} = [B]\{d^{(e)}\} \tag{9.30a}$$

$$\boxed{[B] = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & 0 & \dfrac{\partial f_2}{\partial x} & 0 & \dots & \dots & \dfrac{\partial f_n}{\partial x} & 0 \\ 0 & \dfrac{\partial f_1}{\partial y} & 0 & \dfrac{\partial f_2}{\partial y} & \dots & \dots & 0 & \dfrac{\partial f_n}{\partial y} \\ \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial x} & \dots & \dots & \dfrac{\partial f_n}{\partial y} & \dfrac{\partial f_n}{\partial x} \end{bmatrix}} \tag{9.30b}$$

The matrix $[B]$ in Equation (9.30b) is called the strain–displacement matrix. Substituting Equation (9.30a) into Equation (9.28d), we obtain the strain energy density for an element as shown in Equation (9.31a).

$$U_0^{(e)} = \frac{1}{2}\{d^{(e)}\}^T\{B\}^T[\boldsymbol{E}]\{B\}\{d^{(e)}\} \tag{9.31a}$$

The strain energy for the element can be written as shown in Equation (9.31b) and (9.31c).

$$U^{(e)} = \int_{V^{(e)}} U_0^{(e)} dV = \int_{V^{(e)}} \frac{1}{2}\{d^{(e)}\}^T\{B\}^T[\boldsymbol{E}]\{B\}\{d^{(e)}\}dV = \frac{1}{2}\{d^{(e)}\}^T[K^{(e)}]\{d^{(e)}\} \quad \textbf{(9.31b)}$$

$$\boxed{[K^{(e)}] = \int_{V^{(e)}} [B]^T[\boldsymbol{E}][B]dV} \quad \textbf{(9.31c)}$$

The matrix $[K^{(e)}]$ is the element stiffness matrix.

9.6.1 Constant Strain Triangle

The constant strain triangle (CST) is the simplest element in two dimensions and was one of the first to be used in finite element analysis. As the name suggests, the strain in the element is a constant, which implies that the displacements are linear functions of x and y as shown in Equation (9.32a).

$$u^{(e)} = a_0 + a_1 x + a_2 y \qquad \text{and} \qquad \mathrm{v}^{(e)}(x, y) = b_0 + b_1 x + b_2 y \quad \textbf{(9.32a)}$$

There are three constants for u and v in Equation (9.32a). To evaluate the three constants, we need nodal displacements at three nodes, which define a triangle shown in Figure 9.14. The strains in the element can be determined and are found to be constants, as shown in Equation (9.32b).

$$\varepsilon_{xx}^{(e)} = \frac{\partial u^{(e)}}{\partial x} = a_1 \qquad \varepsilon_{yy}^{(e)} = \frac{\partial \mathrm{v}^{(e)}}{\partial y} = b_2 \qquad \gamma_{xy}^{(e)} = \frac{\partial u^{(e)}}{\partial y} + \frac{\partial \mathrm{v}^{(e)}}{\partial x} = a_2 + b_1 \quad \textbf{(9.32b)}$$

The constants in Equation (9.32a) can be solved in terms of the nodal displacements of the three nodes shown in Figure 9.14 to obtain Equation (9.32c).

$$u^{(e)}(x, y) = \sum_{i=1}^{3} f_i(x, y) u_i^{(e)} \qquad \mathrm{v}^{(e)}(x, y) = \sum_{i=1}^{3} f_i(x, y) \mathrm{v}_i^{(e)} \quad \textbf{(9.32c)}$$

Consider the u displacement at the jth node as given by Equation (9.32d).

$$u^{(e)}(x_j, y_j) = \sum_{i=1}^{3} f_i(x_j, y_j) u_i^{(e)} = u_j^{(e)} \quad \textbf{(9.32d)}$$

If Equation (9.32d) is to be valid, the interpolation functions must satisfy the condition given by Equation (9.32e), which is similar to the property of Lagrange polynomials given by Equation (9.4a).

$$f_i(x_j, y_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \textbf{(9.32e)}$$

The plots of the interpolation functions shown Figure 9.15 are constructed by using the observations that these functions are linear and must satisfy Equation (9.32e).

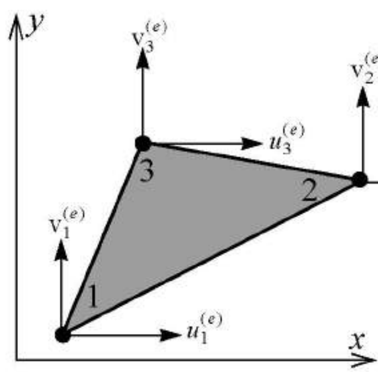

Figure 9.14 Constant strain triangle.

The strain–displacement matrix $[B]$ can be found as shown in Equation (9.33a).

$$[B] = \frac{1}{2A^{(e)}}\begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix} \quad \textbf{(9.33a)}$$

$$A^{(e)} = \frac{1}{2}[(x_2 - x_1)(y_3 - y_1) - (y_1 - y_2)(x_1 - x_3)] \quad \textbf{(9.33b)}$$

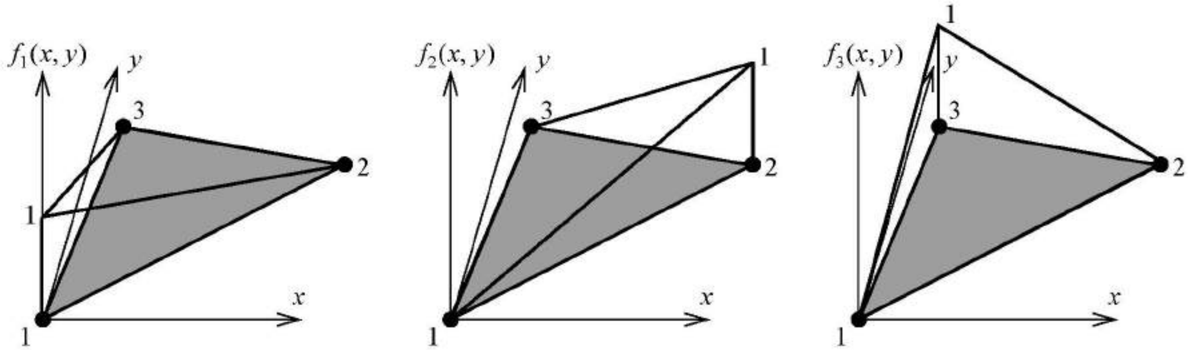

Figure 9.15 Interpolation functions for a CST.

The $A^{(e)}$ in Equation (9.33b) is the area of the triangle. The stiffness matrix can now be constructed from Equation (9.31c). Noting that the matrices $[B]$ and $[\tilde{E}]$ are constants in Equation (9.31c) and that the volume of the triangular element is area $A^{(e)}$ multiplied by the element thickness $t^{(e)}$, we obtain the stiffness matrix given by Equation (9.33c).

$$[K^{(e)}] = [B]^T[\tilde{E}][B]\left[\int_{V^{(e)}} dV\right] = A^{(e)}t^{(e)}[B]^T[\tilde{E}][B] \qquad \textbf{(9.33c)}$$

The load vector can be constructed as discussed in Section 9.3.2, and the solution procedure would then proceed as discussed in Section 9.3.9. There are many issues and concepts in application of the equations described in this section, and the reader is referred to finite element textbooks for additional details.

EXAMPLE 9.4

Obtain the strain–displacement matrix $[B]$ for the constant strain triangle given by Equation (9.33a).

PLAN

Equation (9.32b) shows that the strain expression does not contain a_0 and b_0. The constants a_1 and a_2 in Equation (9.32a) can be found in terms of the nodal displacement in the x direction. From these expressions, by replacing u with v and a's with b's, the constants b_1 and b_2 in Equation (9.32a) can be determined. The strain expressions can be written in matrix form and the strain–displacement matrix $[B]$ determined.

SOLUTION

We use the coordinates of the three nodes substituted in Equation (9.32a) and the displacements equated to the nodal displacements to obtain Equation (E1) through (E3).

$$a_0 + a_1x_1 + a_2y_1 = u_1^{(e)} \qquad \text{(E1)}$$

$$a_0 + a_1x_2 + a_2y_2 = u_2^{(e)} \qquad \text{(E2)}$$

$$a_0 + a_1x_3 + a_2y_3 = u_3^{(e)} \qquad \text{(E3)}$$

We can use Cramer's rule, to find the constants. The determinant $|D|$ of the matrix on the left-hand side of Equation (E1) through (E3) can be written as shown in Equation (E4)

$$|D| = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \qquad \text{(E4)}$$

Row 1 in Equation (E4) can be subtracted from rows 2 and 3 and the determinant evaluated from the first column, as shown in Equation (E5),

$$|D| = \begin{vmatrix} 1 & x_1 & y_1 \\ 0 & (x_2-x_1) & (y_2-y_1) \\ 0 & (x_3-x_1) & (y_3-y_1) \end{vmatrix} = (x_2-x_1)(y_3-y_1)-(y_2-y_1)(x_3-x_1) = 2A \qquad \text{(E5)}$$

where A is the area of the triangle as given in Equation (9.33b).

By Cramer's rule, the constants a_1 and a_2 can be found as shown in Equation (E6) and (E7).

$$a_1 = \frac{1}{|D|}\begin{vmatrix} 1 & u_1^{(e)} & y_1 \\ 1 & u_2^{(e)} & y_2 \\ 1 & u_3^{(e)} & y_3 \end{vmatrix} = \frac{1}{2A}[-u_1^{(e)}(y_3-y_2)+u_2^{(e)}(y_3-y_1)-u_3^{(e)}(y_2-y_1)] \qquad \text{(E6)}$$

$$a_2 = \frac{1}{|D|}\begin{vmatrix} 1 & x_1 & u_1^{(e)} \\ 1 & x_2 & u_2^{(e)} \\ 1 & x_3 & u_3^{(e)} \end{vmatrix} = \frac{1}{2A}[u_1^{(e)}(x_3-x_2) - u_2^{(e)}(x_3-x_1) + u_3^{(e)}(x_2-x_1)] \tag{E7}$$

The constants b_1 and b_2 can be written by replacing the u's with v's as shown in Equation (E8) and (E9).

$$b_1 = \frac{1}{2A}[-\mathrm{v}_1^{(e)}(y_3-y_2) + \mathrm{v}_2^{(e)}(y_3-y_1) - \mathrm{v}_3^{(e)}(y_2-y_1)] \tag{E8}$$

$$b_2 = \frac{1}{2A}[\mathrm{v}_1^{(e)}(x_3-x_2) - \mathrm{v}_2^{(e)}(x_3-x_1) + \mathrm{v}_3^{(e)}(x_2-x_1)] \tag{E9}$$

The strains in the element can be obtained by substituting Equation (E6) through (E9) into (9.32b) to obtain Equation (E10) through (E12).

$$\varepsilon_{xx}^{(e)} = \frac{1}{2A}[u_1^{(e)}(y_2-y_3) + u_2^{(e)}(y_3-y_1) + u_3^{(e)}(y_1-y_2)] \tag{E10}$$

$$\varepsilon_{yy}^{(e)} = \frac{1}{2A}[\mathrm{v}_1^{(e)}(x_3-x_2) + \mathrm{v}_2^{(e)}(x_1-x_3) + \mathrm{v}_3^{(e)}(x_2-x_1)] \tag{E11}$$

$$\gamma_{xy}^{(e)} = \frac{1}{2A}\begin{bmatrix} u_1^{(e)}(x_3-x_2) + u_2^{(e)}(x_1-x_3) + u_3^{(e)}(x_2-x_1) \\ +\mathrm{v}_1^{(e)}(y_2-y_3) + \mathrm{v}_2^{(e)}(y_3-y_1) + \mathrm{v}_3^{(e)}(y_1-y_2) \end{bmatrix} \tag{E12}$$

Equation (E10) through (E12) can be written in matrix form as shown in Equation (E13).

$$\{\tilde{\varepsilon}^{(e)}\} = \begin{Bmatrix} \varepsilon_{xx}^{(e)} \\ \varepsilon_{yy}^{(e)} \\ \gamma_{xy}^{(e)} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} y_2-y_3 & 0 & y_3-y_1 & 0 & y_1-y_2 & 0 \\ 0 & x_3-x_2 & 0 & x_1-x_3 & 0 & x_2-x_1 \\ x_3-x_2 & y_2-y_3 & x_1-x_3 & y_3-y_1 & x_2-x_1 & y_1-y_2 \end{bmatrix}\begin{Bmatrix} u_1^{(e)} \\ \mathrm{v}_1^{(e)} \\ u_2^{(e)} \\ \mathrm{v}_2^{(e)} \\ u_3^{(e)} \\ \mathrm{v}_3^{(e)} \end{Bmatrix} = [B]\{d^{(e)}\} \tag{E13}$$

From Equation (E13), we see that the $[B]$ matrix is as given by Equation (9.33a).

COMMENT

The constants a_0 and b_0 in Equation (9.32a) have no effect on the stiffness matrix and hence none on the nodal displacement values. However, if the displacement at any point inside the element were needed, the values of these constants in terms of the nodal displacements would be needed (see Problem 9.24).

9.7 CLOSURE

In this chapter, we used one-dimensional structural elements to elaborate the procedure for analysis by means of the finite element method. Familiarity with the following concepts will help in reading manuals and documents accompanying finite element software packages: nodes, elements, mesh, discretization, interpolation functions, Lagrange polynomials, Hermite polynomials, element stiffness matrix, global stiffness matrix, element load vector, nodal forces, nodal displacements, mesh refinement, h-method, p-method, and r-method.

The concepts introduced in this book can be further developed in solid mechanics courses covering subjects such as plates and shells, the finite element method, elasticity, plasticity, continuum mechanics, fracture mechanics, the mechanics of composites, and biomechanics.

PROBLEMS

9.1 In a linear axial rod element, the nodal displacements were found to be $u_1^{(1)} = 0.05$ mm and $u_2^{(1)} = 0.25$ mm . The length of the element is 400 mm, the cross-sectional area $A = 50$ mm^2, and the modulus of elasticity $E = 200$ GPa. Determine (a) the displacement from node 1 at 100 mm, (b) the axial stress from node 1 at 100, and (c) the strain energy in the element.

9.2 In a quadratic axial rod element, the nodal displacements at the three equally spaced nodes were found to be

$$u_1^{(1)} = 0.0027 \text{ in} \qquad u_2^{(1)} = 0.0098 \text{ in} \qquad u_3^{(1)} = 0.017 \text{ in}$$

If the length of the element is 6 inches and the modulus of elasticity $E = 10{,}000$ ksi, determine (a) the displacement at 2 inches from node 1 and (b) the stress at 2 inches from node 1.

9.3 In a beam element of length 20 inches, the nodal displacement and slope at the two end nodes were found to be:

$$v_1^{(1)} = -0.407 \text{ in} \qquad v_2^{(1)} = -0.407 \text{ in}$$

$$\theta_1^{(1)} = -0.012 \qquad \theta_2^{(1)} = 0.012$$

Find the deflection and slope at the midpoint of the element.

9.4 For the axial rod shown in Figure 9.8, find the stress at point *B*, the displacement at point *C*, and the strain energy in each element. Use the following FEM model: one linear element in *AB* and one quadratic element in *BC*. Compare your results with those shown in Table 9.1.

9.5 For the axial rod shown in Figure 9.8, find the stress at point *B*, the displacement at point *C*, and the strain energy in each element. Use the following FEM model: two equal linear elements for the entire rod *AC*. Compare your results with those shown in Table 9.1.

9.6 The axial rod shown in Figure P9.6 has an axial rigidity $EA = 15(10^6)$ lb. The rod is to be modeled by using a linear element for *AB* and a linear element for *BC*. Determine the displacement at point *B* and the reaction force at *A*.

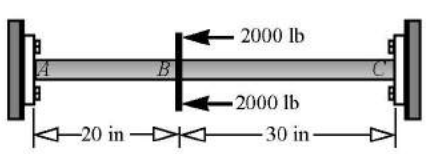

Figure P9.6

9.7 A force $F = 20$ kN is applied to the roller that slides inside a slot as shown in Figure P9.7. Both bars have cross-sectional area $A = 100$ mm^2 and modulus of elasticity $E = 200$ GPa. Bars *AP* and *BP* have lengths of $L_{AP} = 200$ mm and $L_{BP} = 250$ mm, respectively. Determine the displacement of the roller and the axial stress in bar *A*, using linear elements to represent each bar..

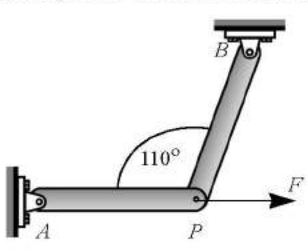

Figure P9.7

9.8 A force $F = 20$ kN is applied to a pin as shown in Figure P9.8. Both bars have a cross-sectional area $A = 100$ mm^2 and a modulus of elasticity $E = 200$ GPa. Bars *AP* and *BP* have lengths of $L_{AP} = 200$ mm and $L_{BP} = 250$ mm, respectively. Determine the displacement of pin *P*, using linear elements to represent each bar.

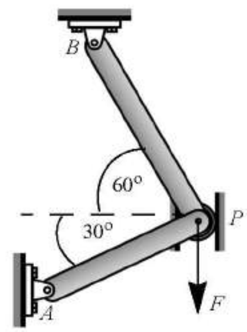

Figure P9.8

9.9 A force $F = 20$ kN is applied to a pin as shown in Figure P9.9. Both bars have a cross-sectional area $A = 100$ mm^2 and a modulus of elasticity $E = 200$ GPa. Bars *AP* and *BP* have lengths of $L_{AP} = 200$ mm and $L_{BP} = 250$ mm, respectively. Determine the displacement of pin *P*, using linear elements to represent each bar..

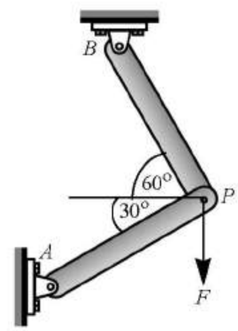

Figure P9.9

9.10 A force $F = 20$ kN is applied to a roller that slides inside a slot as shown in Figure P7.10. Both bars have cross-sectional area $A = 100$ mm^2 and modulus of elasticity $E = 200$ GPa. Bars *AP* and *BP* have lengths of $L_{AP} = 200$ mm and $L_{BP} = 250$ mm, respectively. Determine the displacement of the roller and the axial stress in bar *A*, using linear elements to represent each bar.

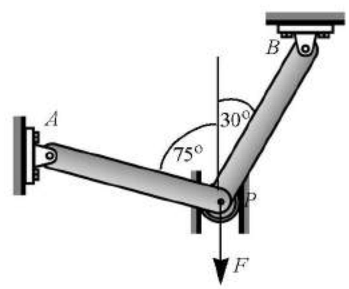

Figure P7.10

ANS. $v_P = -0.3$ mm $\qquad R_{P_x} = 2.89$ kN

9.11 A force $F = 20$ kN is applied to a roller that slides inside a slot as shown in Figure P9.11. Both bars have cross-sectional area $A = 100$ mm^2 and modulus of elasticity $E = 200$ GPa. Bars *AP* and *BP* have lengths of $L_{AP} = 200$ mm and $L_{BP} = 250$ mm, respectively. Determine the displacement of the roller and the axial stress in bar *A*, using linear elements to represent each bar

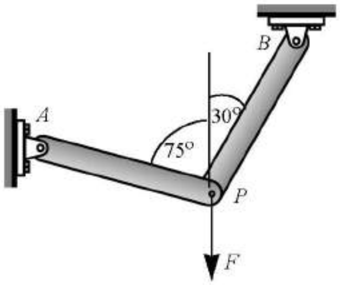

Figure P9.11

9.12 A steel (G_{st} = 12,000 ksi) shaft and a bronze ($G_{Cu/Sn}$ = 5600 ksi) shaft are securely connected at B as shown in Figure P9.12. The diameter of both shaft is 2 in. Determine the maximum torsional shear stress in the entire shaft and the rotation of the section at B, using one linear element to represent each segment, AB and BC..

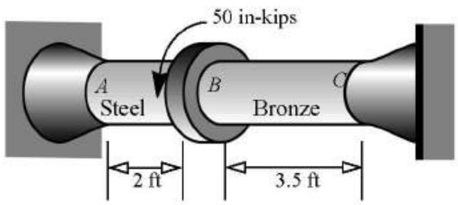

Figure P9.12

ANS. $\phi_B = 0.0516$ rad $\qquad T_A = -40.52$ in · kips

9.13 A solid circular steel (G_{st} = 12,000 ksi, E_{st} = 30,000 ksi) shaft of 4-inch diameter is loaded as shown in Figure P9.13. Determine the rotation of the sections at B and C and the reaction torque at A, using one linear element to represent each segment, AB, BC, and CD.

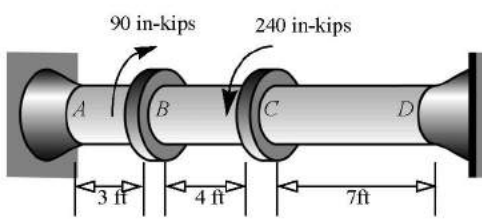

Figure P9.13

9.14 Starting with Equation (7.38c), obtain the first row of the element stiffness matrix given in Equation (9.26a).

9.15 Starting with Equation (7.38c), obtain the second row of the element stiffness matrix given in Equation (9.26a).

9.16 Starting with Equation (7.38c), obtain the third row of the element stiffness matrix given in Equation (9.26a).

9.17 Starting with Equation (7.38c), obtain the fourth row of the element stiffness matrix given in Equation (9.26a).

9.18 Starting with Equation (7.38d), obtain the element load vector given in Equation (9.26a).

9.19 Using just one beam element, determine the deflection and slope at the free end of the beam shown in Figure P9.19. Assume that EI is constant for the beam.

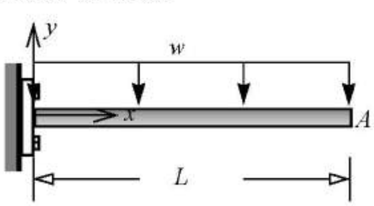

Figure P9.19

9.20 Using just one beam element, determine the deflection and slope at B in the beam shown in Figure P7.20. Assume that EI is constant for the beam.

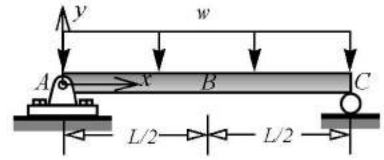

Figure P7.20

ANS. $v_B = -\left(\frac{wL^4}{96EI}\right) \qquad \theta_B = 0$

9.21 Using two beam elements, determine the deflection and slope at the free end and reaction force at B in the beam shown in Figure P9.21. Assume that EI is constant for the beam.

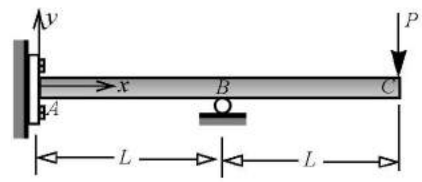

Figure P9.21

ANS. $v_C = -\left(\frac{7PL^3}{12EI}\right) \qquad \theta_C = -\left(\frac{3PL^2}{4EI}\right) \quad R_B = \frac{5P}{2}$

9.22 Using one element, determine the deflection and slope at midpoint of the beam shown in Figure P9.22. Assume that EI is constant for the beam.

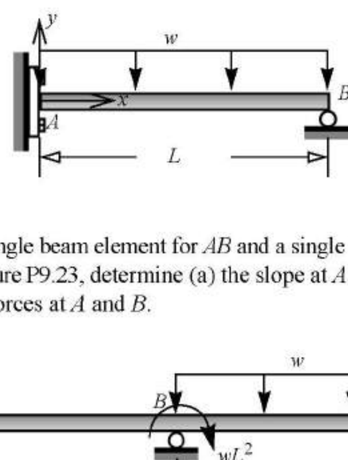

Figure P9.22

9.23 Using a single beam element for AB and a single beam element for BC in Figure P9.23, determine (a) the slope at A and B and (b) the reaction forces at A and B.

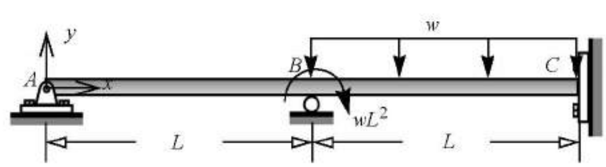

Figure P9.23

9.24 Determine the constants a_0 and b_0 in Equation (9.32a).

9.25 In a constant strain triangle, the continuity of the displacement at the nodes also ensures continuity across the line joining the nodes. To prove this statement, show that the displacement[9] along the line joining nodes 1 and 2 depends only upon the nodal values of nodes 1 and 2.