Plates

A plate is a flat solid body whose thickness is small compared to the other dimensions and is subjected to bending loads.

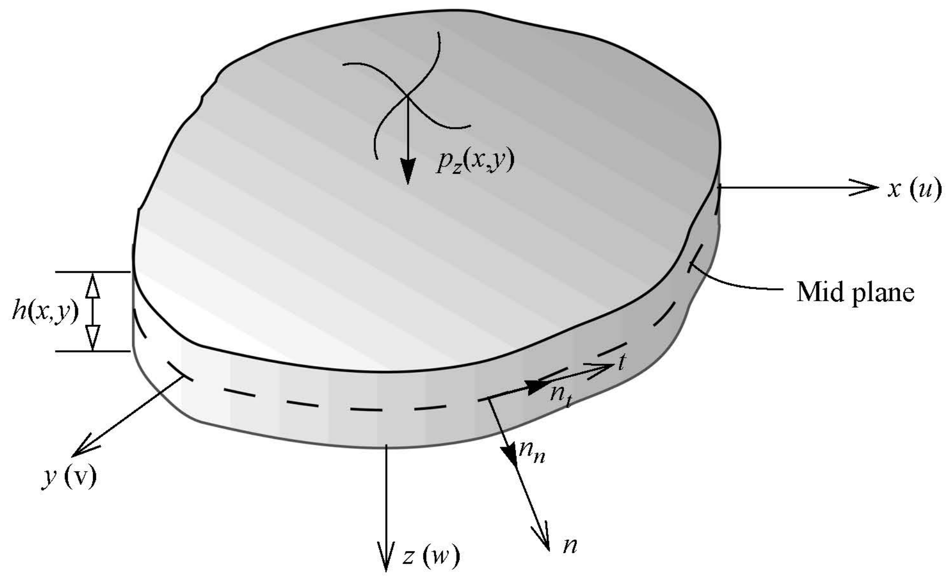
- Examples: Floors, ceilings, windows, disc brakes, ship decks, truck beds

The learning objectives

- Understand the theory of thin plate bending, its limitations, and its applications in design and analysis.
- Understand how to incorporate complexities in plate theory.

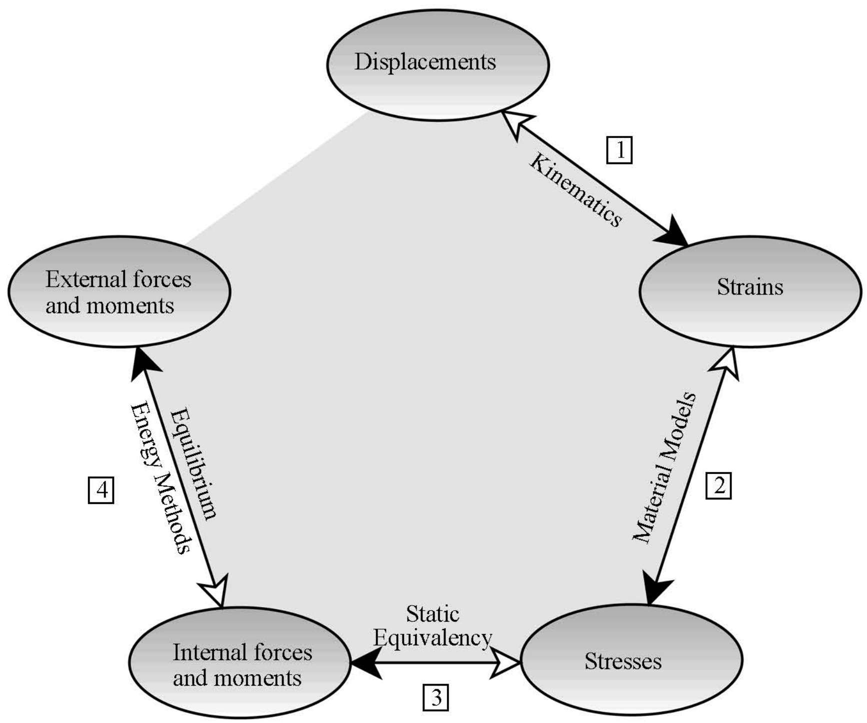
Limitations

1. The mid plane of the plate is flat.
2. The thickness of the plate $h(x,y)$ is an order of magnitude smaller than the dimensions in the x or y direction.
3. The thickness of the plate $h(x,y)$ and transverse loading $p_z(x,y)$ vary *gradually*.
4. We are away from the regions of stress concentration.
5. No in-plane loads-- pure bending.
6. $max|\sigma_{zz}|$ and « $max\{|\sigma_{xx}|, |\sigma_{yy}|, |\tau_{xy}|\}$
7. The loads do not vary with time, or vary so slowly (quasi static problem)
8. We restrict our selves to linear theory.
   Kirchhoff-Love plate theory: $0 \le w \le 0.2h$ ----Linear
   Von-Karman plate theory $0.3h \le w \le h$ ----- Non-linear theory.
   Membrane theory: $w \ge 2h$.



Thin plate theory

- Our objective is to obtain equations relating the bending stresses σ_{xx}, σ_{yy}, and τ_{xy} to internal forces and moments and obtain the boundary value problem governing the transverse deflection *w* of the plate



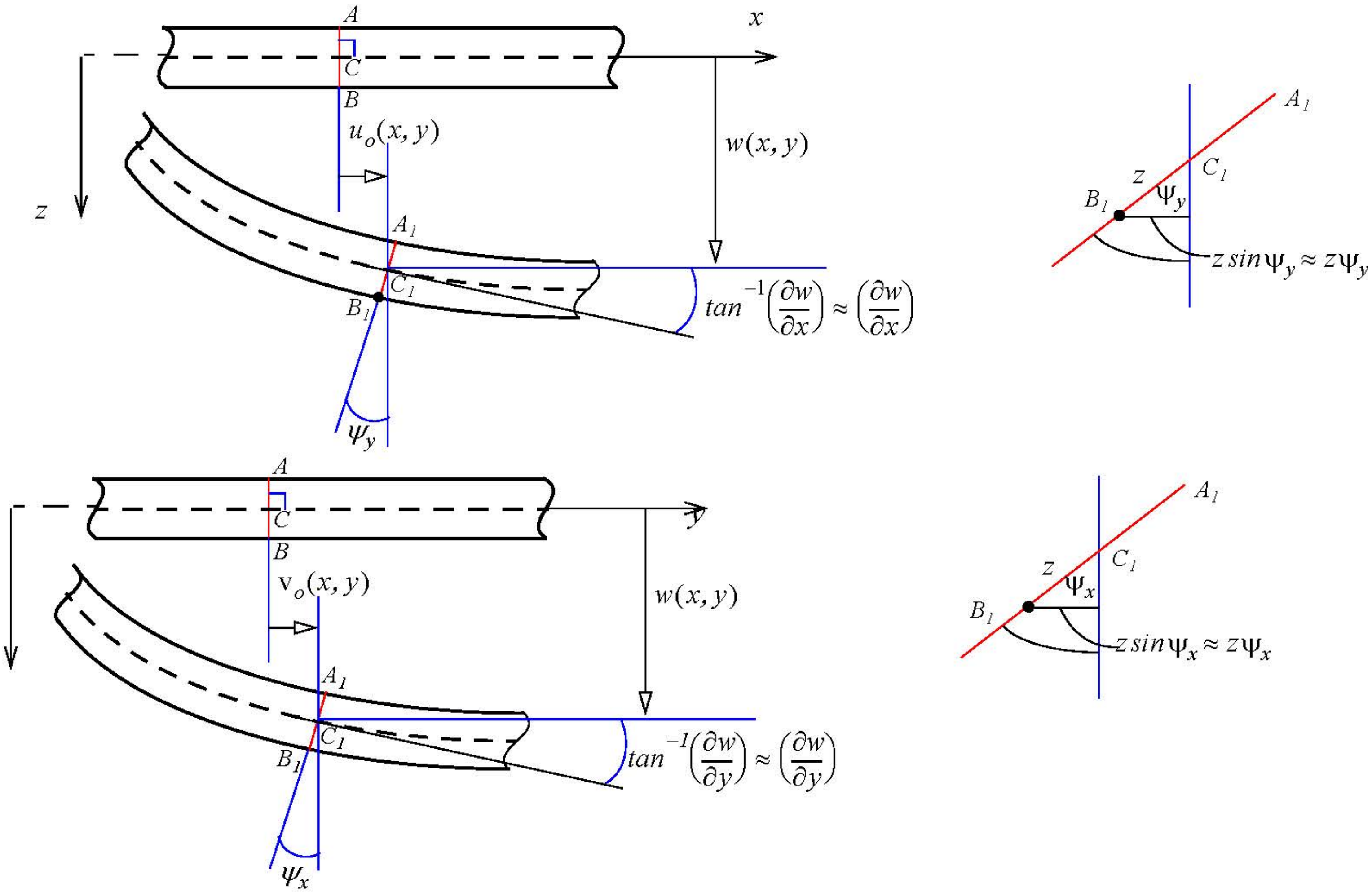
Assumption 1 Deformations are not functions of time.

Assumption 2 Squashing— dimensional changes in the *z* direction, is significantly smaller than bending.

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \approx 0 \qquad \Rightarrow \qquad u = u(x, y, z) \qquad \mathrm{v} = \mathrm{v}(x, y, z) \qquad w = w(x, y)$$

Assumption 3 Plane sections before deformation remain planes after deformation.

- Displacements u and v are linear function of z.



$$u \approx u_o(x, y) - z\,sin\psi_y \qquad v \approx v_o(x, y) - z\,sin\psi_x \qquad w \approx w(x, y)$$

The rotations ψ_x and ψ_y are not functions z. $u_o(x, y) = 0$ $\quad v_o(x, y) = 0$ $\quad$ If there are no inplane forces.

Assumption 4 Plane sections perpendicular to the plate mid surface remain nearly perpendicular after deformation.

The right angle at C remains a right angle at C_1. $\psi_y = tan^{-1}\left(\frac{\partial w}{\partial x}\right)$ $\qquad \psi_x = tan^{-1}\left(\frac{\partial w}{\partial y}\right)$

Assumption 5 Deflection and its derivatives are small.

$$sin\psi_x \approx \psi_x \qquad \text{and} \qquad sin\psi_x \approx \psi_x \qquad\qquad tan^{-1}\left(\frac{\partial w}{\partial x}\right) \approx \left(\frac{\partial w}{\partial x}\right) \qquad \text{and} \qquad tan^{-1}\left(\frac{\partial w}{\partial y}\right) \approx \left(\frac{\partial w}{\partial y}\right)$$

$$u \approx -z\left(\frac{\partial w}{\partial x}\right) \qquad v \approx -z\left(\frac{\partial w}{\partial y}\right) \qquad w \approx w(x, y) \qquad \Rightarrow \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \approx 0 \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \approx 0$$

Assumption 6 Strains are small

$$\boxed{\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \qquad \varepsilon_{xx} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}}$$

- The strains vary linearly with z.
- The maximum strain will be either on the top or bottom surface of the plate.
- Curvatures of deformed curved plate are $\partial^2 w / \partial x^2$, $\partial^2 w / \partial y^2$, and $\partial^2 w / \partial x \partial y$.

Material Model

Assumption 7 The material is isotropic.

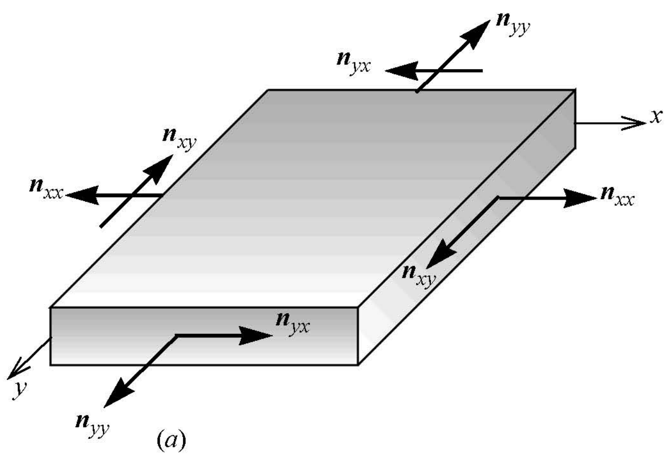
Assumption 8 The material is linearly elastic.

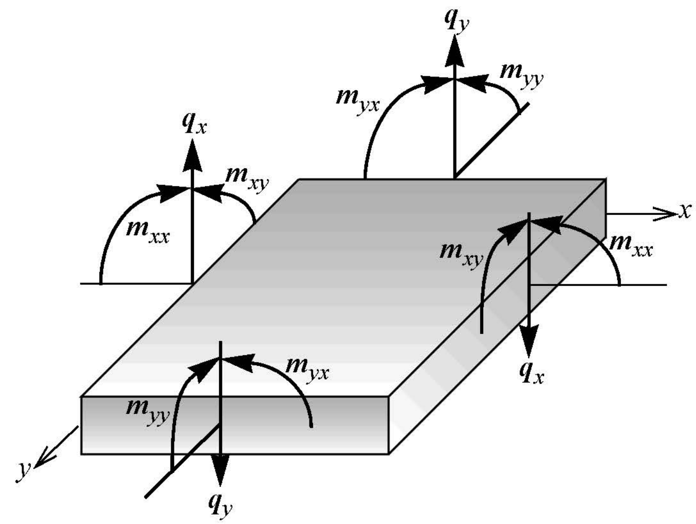
Assumption 9 There are no thermal or non-mechanical strains.

- The plate is in state of *plane stress*. $\sigma_{xx} = \frac{E}{(1-\nu^2)}(\varepsilon_{xx} + \nu\varepsilon_{yy})$ $\qquad \sigma_{yy} = \frac{E}{(1-\nu^2)}(\varepsilon_{yy} + \nu\varepsilon_{xx})$ $\qquad \tau_{xy} = G\gamma_{xy}$

$$\sigma_{xx} = \frac{-Ez}{(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \qquad \sigma_{yy} = \frac{-Ez}{(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \qquad \tau_{xy} = -2Gz\frac{\partial^2 w}{\partial x \partial y}$$

Static Equivalency





$$n_{xx} = \int_h \sigma_{xx} dz \qquad n_{yy} = \int_h \sigma_{yy} dz \qquad n_{xy} = n_{yx} = \int_h \tau_{xy} dz$$

$$m_{xx} = \int_h z\sigma_{xx} dz \qquad m_{yy} = \int_h z\sigma_{yy} dz \qquad m_{xy} = m_{yx} = \int_h z\tau_{xy} dz \qquad q_x = \int_h \tau_{xz} dz \qquad q_y = \int_h \tau_{yz} dz$$

- All *force stress resultants* will have the *units of force per unit length*
- All *moment stress resultants* will have the *units of moments per unit length*.
- The internal shear forces q_x and q_y are necessary for force equilibrium in the z direction, which implies that the stresses τ_{xz} and τ_{yz} cannot be zero but must be small.
- Average values of these stresses can be found by dividing the internal shear forces q_x and q_y by the thickness.
- The maximum of these average values must be an order of magnitude smaller than the maximum inplane stresses.

$$\boxed{max|\tau_{xz}| \text{ and } max|\tau_{yz}| \ll max\{|\sigma_{xx}|, |\sigma_{yy}|, |\tau_{xy}|\}}$$

Location of neutral surface (origin)

$$n_{xx} = \frac{\partial^2 w}{\partial x^2}\int_h \frac{Ez}{(1-\nu^2)}dz + \frac{\partial^2 w}{\partial y^2}\int_h \frac{E\nu z}{(1-\nu^2)}dz = 0 \qquad n_{yy} = \frac{\partial^2 w}{\partial y^2}\int_h \frac{Ez}{(1-\nu^2)}dz + \frac{\partial^2 w}{\partial x^2}\int_h \frac{E\nu z}{(1-\nu^2)}dz = 0 \qquad n_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}\int_h Gz dz = 0$$

Assumption 10 The material is homogeneous across the thickness of the plate

$$n_{xx} = \left[\frac{\partial^2 w}{\partial x^2}\frac{E}{(1-\nu^2)} + \frac{\partial^2 w}{\partial y^2}\frac{E\nu}{(1-\nu^2)}\right]\int_h z dz = 0 \qquad n_{yy} = \left[\frac{\partial^2 w}{\partial y^2}\frac{E}{(1-\nu^2)} + \frac{\partial^2 w}{\partial x^2}\frac{E\nu}{(1-\nu^2)}\right]\int_h z dz = 0 \qquad n_{xy} = -2G\frac{\partial^2 w}{\partial x \partial y}\int_h z dz$$

- $\boxed{\int_h z dz = 0}$; The origin and the neutral surface must be at the mid surface for pure bending of plates.

Stress formulas

$$m_{xx} = -\left(\frac{\partial^2 w}{\partial x^2}\right)\int_h \frac{Ez^2}{(1-\nu^2)}dz - \left(\frac{\partial^2 w}{\partial y^2}\right)\int_h \frac{E\nu z^2}{(1-\nu^2)}dz \qquad m_{yy} = -\left(\frac{\partial^2 w}{\partial y^2}\right)\int_h \frac{Ez^2}{(1-\nu^2)}dz - \left(\frac{\partial^2 w}{\partial x^2}\right)\int_h \frac{E\nu z^2}{(1-\nu^2)}dz \qquad m_{xy} = -2\left(\frac{\partial^2 w}{\partial x \partial y}\right)\int_h G z^2 dz$$

As per Assumption 10 of homogeneity across the thickness, we obtain

$$m_{xx} = -\left[\frac{E}{(1-\nu^2)}\right]\left[\left(\frac{\partial^2 w}{\partial x^2}\right) + \nu\left(\frac{\partial^2 w}{\partial y^2}\right)\right]\int_h z^2 dz \qquad m_{yy} = -\left[\frac{E}{(1-\nu^2)}\right]\left[\left(\frac{\partial^2 w}{\partial y^2}\right) + \nu\left(\frac{\partial^2 w}{\partial x^2}\right)\right]\int_h z^2 dz \qquad m_{xy} = -2G\left(\frac{\partial^2 w}{\partial x \partial y}\right)\int_h z^2 dz$$

$$\int_h z^2 dz = \int_{-h/2}^{h/2} z^2 dz = h^3/12$$

Moment-curvature equations

$$\boxed{m_{xx} = -D\left[\left(\frac{\partial^2 w}{\partial x^2}\right) + \nu\left(\frac{\partial^2 w}{\partial y^2}\right)\right] \qquad m_{yy} = -D\left[\left(\frac{\partial^2 w}{\partial y^2}\right) + \nu\left(\frac{\partial^2 w}{\partial x^2}\right)\right] \qquad m_{xy} = -D(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right) \qquad D = \frac{Eh^3}{12(1-\nu^2)}}$$

- D is called the *bending rigidity of the plate*.

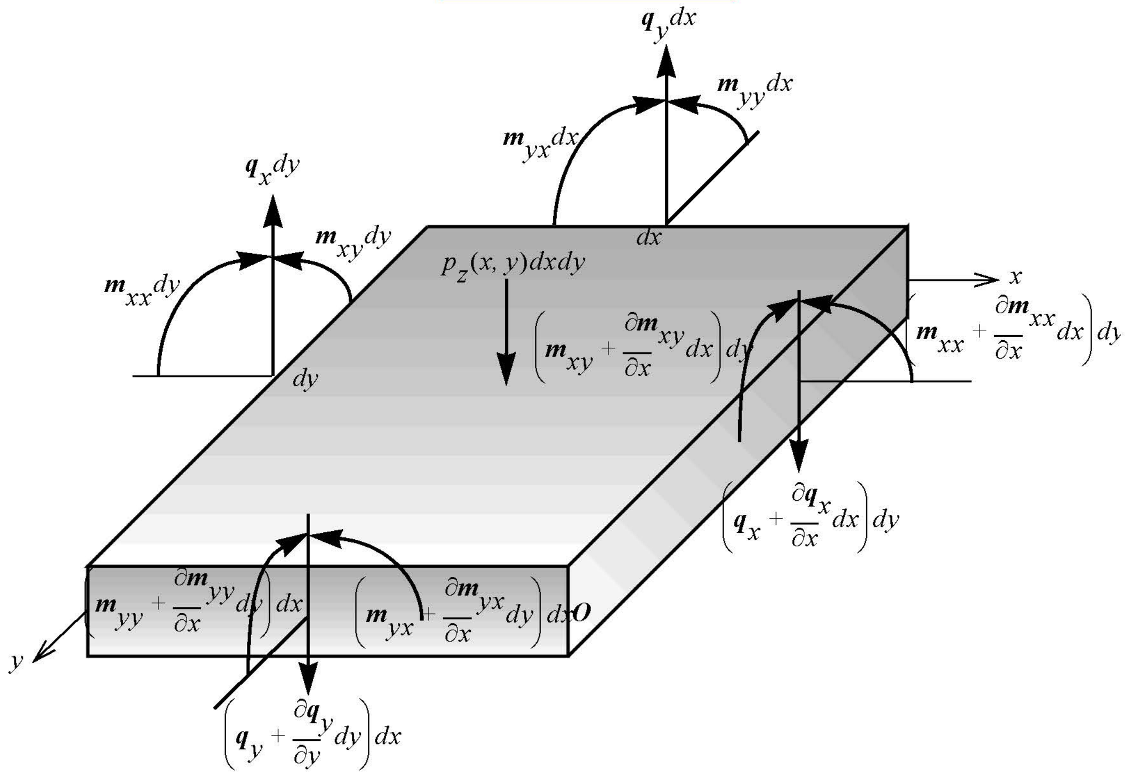
$$\boxed{\sigma_{xx} = \frac{m_{xx} z}{(h^3/12)} \qquad \sigma_{yy} = \frac{m_{yy} z}{(h^3/12)} \qquad \tau_{xy} = \frac{m_{xy} z}{(h^3/12)}}$$

- The bending stresses in plates vary linearly through the thickness and will be maximum at top and bottom surface of the plate.

Equilibrium

Force equilibrium in z-direction:

$$\boxed{\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = -p_z(x, y)}$$



Equilibrium of moment about the x and y direction at point O.

$$\boxed{q_x = \frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{yx}}{\partial y} \qquad q_y = \frac{\partial m_{xy}}{\partial x} + \frac{\partial m_{yy}}{\partial y}}$$

Differential equation

$$q_x = -\frac{\partial}{\partial x}\left[D\left\{\left(\frac{\partial^2 w}{\partial x^2}\right)+\nu\left(\frac{\partial^2 w}{\partial y^2}\right)\right\}\right]-\frac{\partial}{\partial y}\left[\left\{D(1-\nu)\left(\frac{\partial^2 w}{\partial x\partial y}\right)\right\}\right]$$

$$q_y = -\frac{\partial}{\partial x}\left[\left\{D(1-\nu)\left(\frac{\partial^2 w}{\partial x\partial y}\right)\right\}\right]-\frac{\partial}{\partial y}\left[D\left\{\left(\frac{\partial^2 w}{\partial y^2}\right)+\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\right]$$

$$\frac{\partial^2}{\partial x^2}\left[D\left\{\left(\frac{\partial^2 w}{\partial x^2}\right)+\nu\left(\frac{\partial^2 w}{\partial y^2}\right)\right\}\right]+2\frac{\partial^2}{\partial x\partial y}\left[\left\{D(1-\nu)\left(\frac{\partial^2 w}{\partial x\partial y}\right)\right\}\right]+\frac{\partial^2}{\partial y^2}\left[D\left\{\left(\frac{\partial^2 w}{\partial y^2}\right)+\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\right\}\right] = p_z(x,y)$$

Assumption 11 The plate is homogeneous in the x and y direction.

Assumption 12 The plate is of uniform thickness.

h, E, ν, and D are constant.

$$\boxed{q_x = -D\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right] = -D\frac{\partial}{\partial x}(\nabla^2 w)} \qquad \boxed{q_y = -D\frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right] = -D\frac{\partial}{\partial y}(\nabla^2 w)}$$

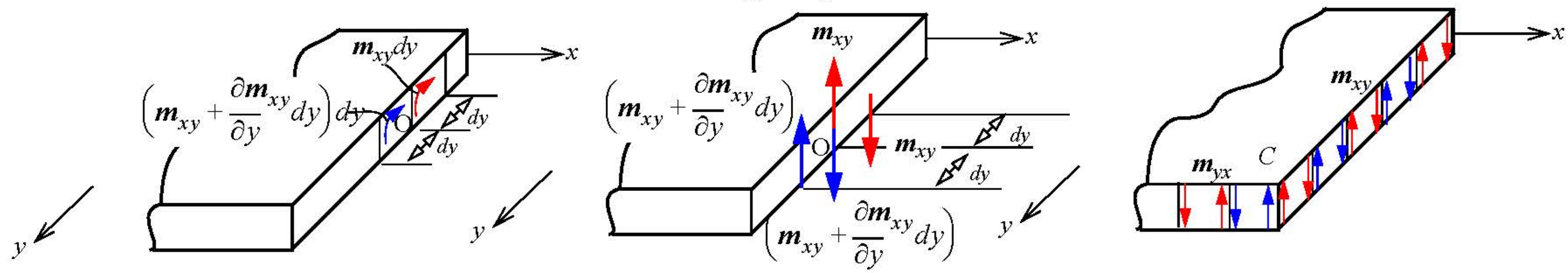
$$\boxed{D\nabla^4 w = D\left[\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right] = p_z(x,y)}$$

where, $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the Laplace operator and $\nabla^4 = \nabla^2\nabla^2$ is the *bi-harmonic* operator.

Boundary Conditions

Fourth order partial differential equation requires two conditions at each boundary point.On boundary point there are three internal quantities, a shear force and two moments. These three quantities need to be reduced to two.

Kirchhoff's arguments

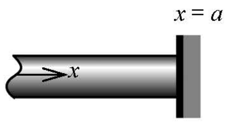
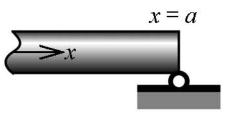
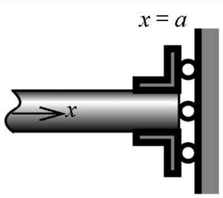
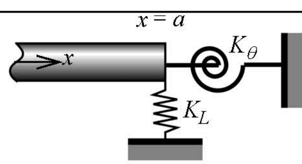


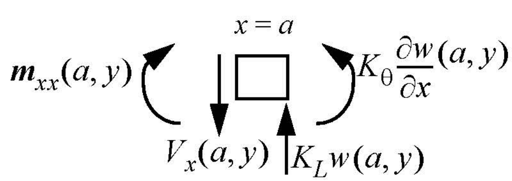
$$V_x = q_x + \frac{\partial m_{xy}}{\partial y} = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right] \qquad V_y = q_y + \frac{\partial m_{yx}}{\partial x} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial y \partial x^2}\right]$$

Corner Force

$$R_{corner} = m_{xy} + m_{yx} = 2m_{xy} = -2D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

On x = constant	Specify $[V_x \text{ or } w]$ and $\left[m_{xx} \text{ or } \frac{\partial w}{\partial x}\right]$	On a corner: Specify $[R_{corner} \text{ or } w]$
On y = constant	Specify $[V_y \text{ or } w]$ and $\left[m_{yy} \text{ or } \frac{\partial w}{\partial y}\right]$	

Type of support		Boundary Condition: Specify [V_x or w]	Boundary Condition: Specify $\left[\boldsymbol{m}_{xx} \text{ or } \frac{\partial w}{\partial x}\right]$
Clamped		$w(a, y) = 0$	$\frac{\partial w}{\partial x}(a, y) = 0$
Simply Supported		$w(a, y) = 0$	$\boldsymbol{m}_{xx}(a, y) = 0$ or $\frac{\partial^2 w}{\partial x^2}(a, y) = 0$
Free		$V_x(a, y) = 0$ or $\left.\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial y \partial x^2}\right]\right\vert_{x=a} = 0$	$\boldsymbol{m}_{xx}(a, y) = 0$ or $\left.\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]\right\vert_{x=a}$
Roller		$V_x(a, y) = 0$ or $\left.\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial y \partial x^2}\right]\right\vert_{x=a} = 0$	$\frac{\partial w}{\partial x}(a, y) = 0$
Elastic		$V_x(a, y) = 0 = K_L w(a, y)$	$\boldsymbol{m}_{xx}(a, y) = K_\theta \frac{\partial w}{\partial x}(a, y)$



C5.1 The cross section of laminated plate made from two materials is shown below. Both materials have the same Poisson's ratio but different modulus of elasticity. The displacement field is given

$$u \approx -(z - z_o)(\partial w / \partial x) \qquad \mathrm{v} \approx -(z - z_o)(\partial w / \partial y) \qquad w \approx w(x, y)$$

where, z is measured from the mid surface and z_o is the location of the neutral surface. (a) Determine the value of z_o assuming all assumptions except for material homogeneity across the thickness are valid. (b) Obtain stress formulas and the differential equation governing the deflection of the laminated plate. (c) By substituting $E_1 = E_2$, show that the results for parts a and b give the same results as classical plate theory for homogeneous material.

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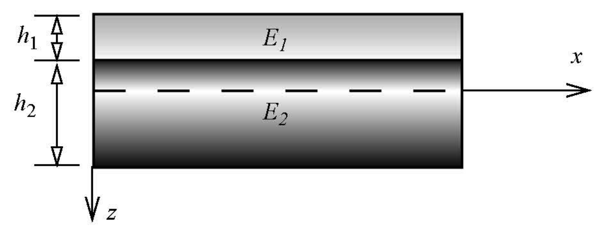


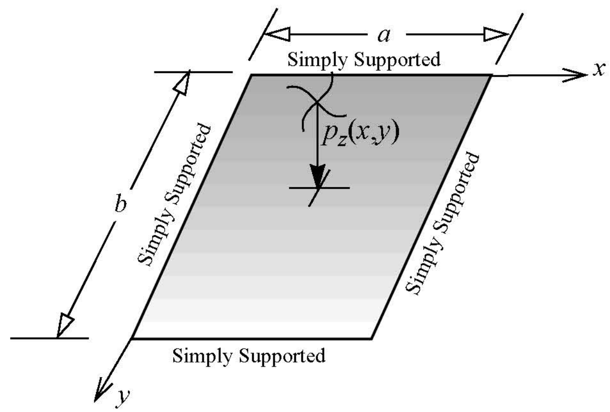
C5.2 In *Mindlin-Reissner* plate theory Assumption 4 of planes sections perpendicular to the plate mid surface remain nearly perpendicular after deformation is dropped to account for shear. (*a*) Starting with the displacement field below obtain the stress formulas and differential equations if all assumptions except Assumption 4 are valid.

$$u \approx -z\psi_x(x, y) \qquad \text{v} \approx -z\psi_y(x, y) \qquad w \approx w(x, y)$$

(*b*) Show the results reduce to those of classical plate theory by substituting $\psi_x = \partial w / \partial x$ and $\psi_y = \partial w / \partial y$.

Navier's solution of rectangular plates

In 1820, *Navier* found the solution for a rectangular plate that is simply supported on all its boundary.



Differential Equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_z(x, y)}{D}$$

Boundary Conditions

On $x = 0$ $\quad w(0, y) = 0 \quad \dfrac{\partial^2 w(0, y)}{\partial x^2} = 0$ $\qquad$ On $x = a$ $\quad w(a, y) = 0 \quad \dfrac{\partial^2 w(a, y)}{\partial x^2} = 0$

On $y = 0$ $\quad w(x, 0) = 0 \quad \dfrac{\partial^2 w(x, 0)}{\partial y^2} = 0$ $\qquad$ On $y = b$ $\quad w(x, b) = 0 \quad \dfrac{\partial^2 w(x, b)}{\partial y^2} = 0$

- Only even derivatives in the boundary value problem.
- Even derivatives of Sine (Cosines) functions produce Sine (Cosine) functions.
- Sine functions can satisfy all boundary conditions.

$$w(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin\left(m\pi\frac{x}{a}\right)\sin\left(n\pi\frac{y}{b}\right) \qquad p_z(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P_{mn} \sin\left(m\pi\frac{x}{a}\right)\sin\left(n\pi\frac{y}{b}\right)$$

where, W_{mn} and P_{mn} are constant coefficients to be determined.

- The series consists of independent functions.
- The infinite series is complete *for this class of problem*.

Substituting $w(x, y)$ and $p_z(x, y)$ we obtain

$$W_{mn} = \frac{1}{D\pi^4}\frac{P_{mn}}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

To determine the constants P_{mn} we use the orthogonality conditions of Sine functions given below.

orthogonality condition: $$\int_0^{\pi} \sin(m\theta)\sin(n\theta)d\theta = \begin{cases} 0 & m \neq n \\ \dfrac{\pi}{2} & m = n \end{cases}$$

$$P_{mn} = \frac{4}{ab}\int_0^b\int_0^a p_z(x, y)\sin\left(m\pi\frac{x}{a}\right)\sin\left(n\pi\frac{y}{b}\right)dxdy$$

Uniform distributed load $p_y(x,y) = p_o$

$$P_{mn} = \begin{cases} 0 & m,n = 2,4,6.\cdots \\ \dfrac{16p_o}{\pi^2 mn} & m,n = 1,3,5\cdots \end{cases} \qquad W_{mn} = \begin{cases} 0 & m,n = 2,4,6.\cdots \\ \dfrac{16p_o}{D\pi^6 mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2} & m,n = 1,3,5\cdots \end{cases}$$

$$w(x,y) = \frac{16p_o}{D\pi^6} \sum_{m=1,3,5.\cdots}^{\infty} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2} \sin\left(m\pi\frac{x}{a}\right) \sin\left(n\pi\frac{y}{b}\right)$$

$$\boldsymbol{m}_{xx}(x,y) = \left(\frac{16p_o}{\pi^4}\right) \sum_{m=1,3,5.\cdots}^{\infty} \sum_{n=1,3,5\cdots}^{\infty} \frac{\left[\left(\frac{m}{a}\right)^2 + \nu\left(\frac{n}{b}\right)^2\right]}{mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2} \sin\left(m\pi\frac{x}{a}\right) \sin\left(n\pi\frac{y}{b}\right)$$

Maximum values of w and $\boldsymbol{m}_{xx}$ at $x = a/2$ and $y = b/2$ ----Center of plate

Convergence Study

Square plate: $b = a$ Poisson's ratio of $\nu = 1/3$.

$$w_{max} = \frac{16p_o a^4}{D\pi^6} \sum_{m=1,3,5.\cdots}^{\infty} \sum_{n=1,3,5\cdots}^{\infty} \frac{(-1)^{(m+n-2)/2}}{mn[m^2+n^2]^2}$$

$$(\boldsymbol{m}_{xx})_{max} = \left(\frac{16p_o a^2}{\pi^4}\right) \sum_{m=1,3,5.\cdots}^{\infty} \sum_{n=1,3,5\cdots}^{\infty} \frac{[m^2+\nu n^2](-1)^{(m+n-2)/2}}{mn[m^2+n^2]^2}$$

$$\% \text{ difference } = \frac{T_n - T_{16}}{T_{16}} \times 100 \text{ where } T_n \text{ is the value after } n \text{ terms.}$$

Convergence Study

No. of terms	m	n	$\frac{w_{max}(10^{-3})}{(p_o a^4)/D}$		$\frac{(m_{xx})_{max}(10^{-3})}{p_o a^2}$	
			value	% difference	value	% difference
1	1	1	4.1606	-2.4278	54.7519	-11.8235
2	3	1	4.1052	-1.0621	49.6417	-1.3867
3	1	3	4.0497	0.3036	47.4517	3.0863
4	3	3	4.0554	0.1631	48.1276	1.7057
5	5	1	4.0603	0.0419	49.3587	-0.8087
6	1	5	4.0653	-0.0793	49.8123	-1.7350
7	5	3	4.0643	-0.0557	49.5470	-1.1933
8	3	5	4.0633	-0.0321	49.3828	-0.8580
9	5	5	4.0636	-0.0386	49.4705	-1.0369
10	7	1	4.0626	-0.0152	49.0074	-0.0912
11	1	7	4.0617	0.0082	48.8447	0.2411
12	7	3	4.0619	0.0024	48.9656	-0.0058
13	3	7	4.0622	-0.0034	49.0245	-0.1261
14	7	5	4.0621	-0.0013	48.9754	-0.0258
15	5	7	4.0620	0.0009	48.9400	0.0466
16	7	7	4.0620	0.0000	48.9628	0.0000

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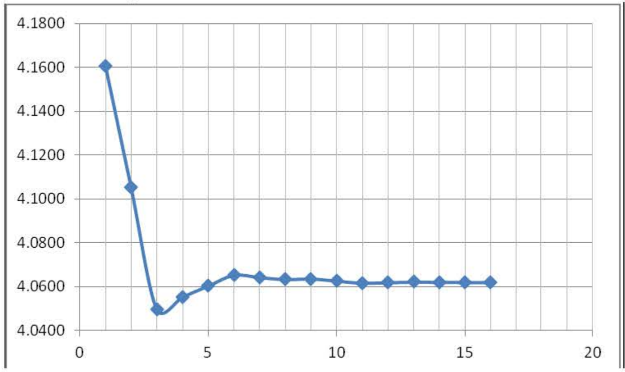
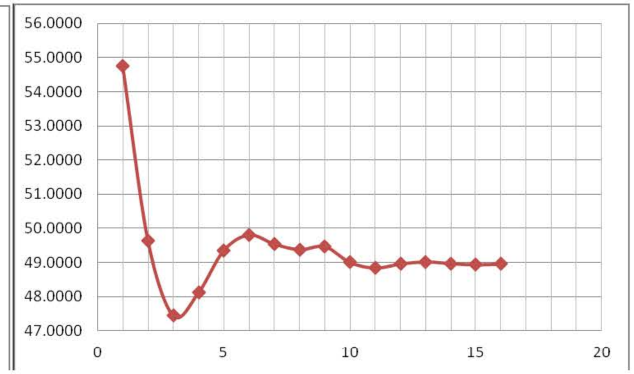
$$\frac{w_{max}(10^{-3})}{(p_o a^4)/D} \text{ vs. number of terms}$$



$$\frac{(\boldsymbol{m}_{xx})_{max}(10^{-3})}{p_o a^2} \text{ vs. number of terms}$$



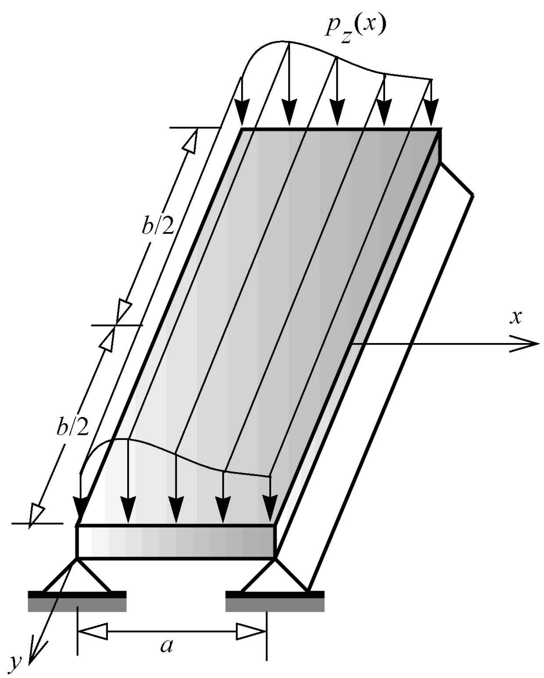
• *differentiation increases and integration decreases the error of approximation.*

Nadai-Levy solution of rectangular plates

1. Plate is simply supported on two opposite sides as shown below. The other two sides can have any type of support or boundary conditions. The boundary conditions at $x = 0$ and $x = a$ can be written as

$$w(0,y) = 0 \qquad \frac{\partial^2 w}{\partial x^2}(0,y) = 0 \qquad w(a,y) = 0 \qquad \frac{\partial^2 w}{\partial x^2}(a,y) = 0 \tag{5.1a}$$

2. The loading p_z is only dependent on x, that is, $p_z(x,y) = p_z(x)$.



- the deflection solution in two parts: $w(x,y) = w_p(x) + w_h(x,y)$

$$\frac{d^4 w_p}{dx^4} = \frac{p_z(x)}{D} \qquad \frac{\partial^4 w_h}{\partial x^4} + 2\frac{\partial^4 w_h}{\partial x^2 \partial y^2} + \frac{\partial^4 w_h}{\partial y^4} = 0 \tag{5.2a}$$

Homogeneous solution

$$w_h(x, y) = \sum_{m=1}^{\infty} H_m(y) sin(m\pi x/a) \tag{5.2b}$$

Substituting (5.2c)

$$\left(\frac{m\pi}{a}\right)^4 H_m - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 H_m}{dy^2} + \frac{d^4 H_m}{dy^4} = 0 \tag{5.2d}$$

The homogeneous solution is

$$\boxed{w_h(x, y) = \sum_{m=1}^{\infty} \{[A_m + D_m(m\pi y/a)] cosh(m\pi y/a) + [C_m + B_m(m\pi y/a)] sinh(m\pi y/a)\} sin(m\pi y/a)}$$

Particular solution

Method I:

$$w_p(x) = \sum_{m=1}^{\infty} W_m sin(m\pi x/a) \qquad p_z(x) = \sum_{m=1}^{\infty} P_m sin(m\pi x/a)$$

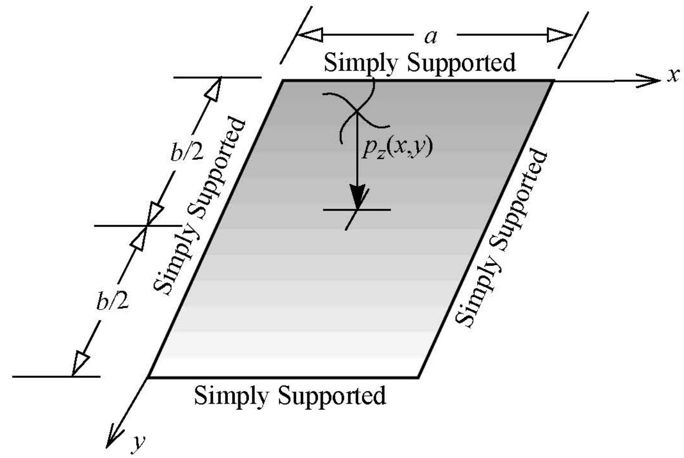
$$\boxed{W_m = \left(\frac{a}{m\pi}\right)^4 \frac{P_m}{D} \qquad P_m = \frac{2}{a}\int_0^a p_z(x) sin(m\pi x/a)dx}$$

Method II: Direct integration for given value of p_z

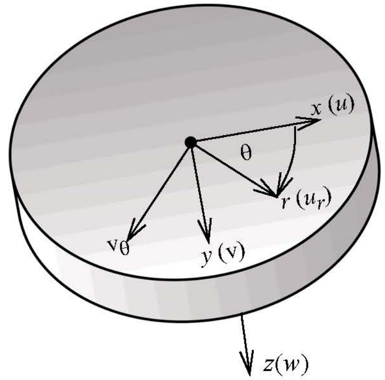
For uniform load $p_z = p_0$.

$$w_p = \frac{p_0 a^4}{24D}\left[\left(\frac{x}{a}\right)^4 - 2\left(\frac{x}{a}\right)^3 + \left(\frac{x}{a}\right)\right]$$

C5.3 The simply supported plate shown below is subject to a uniform distributed force of p_o. (a) Obtain a series solution for deflection w using Nadai-Levy Method. (b) For maximum deflection w and bending moment $\boldsymbol{m}_{xx}$, compare the convergence of Nadai-Levy Method with Navier's method for a square plate and Poisson's ratio of. $\nu = 1/3$



Circular plates



Axisymmetric: Loading, material property, and geometry are independent of the angular coordinate q.

Solution is independent of angular coordinate θ

$$tw = w(r) \qquad \partial w / \partial\theta = 0 \qquad \partial w / \partial r = dw / dr$$

1. Displacements

$= -z\left(\frac{\partial w}{\partial r}\right) \quad v_\theta = -\frac{z}{r}\left(\frac{\partial w}{\partial \theta}\right) \quad w \approx w(r$	$u_r = -z\frac{dw}{dr} \quad v_\theta = 0 \quad w \approx w(r)$

2. Strains

$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = -z\left(\frac{\partial^2 w}{\partial r^2}\right)$	$\varepsilon_{rr} = -z\frac{d^2 w}{dr^2}$
$\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = -z\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)$	$\varepsilon_{\theta\theta} = -\frac{z}{r}\frac{dw}{dr}$
$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} = -2z\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right)$	$\gamma_{r\theta} = 0$

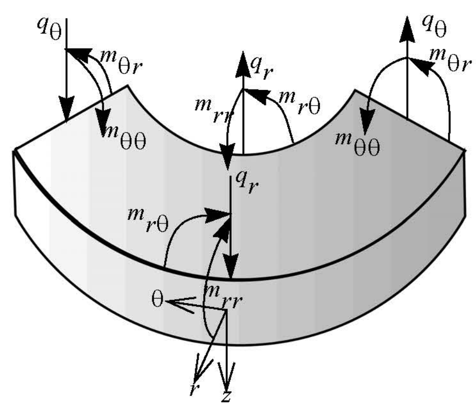
3. Stresses

$$\sigma_{rr} = \frac{E}{(1-\nu^2)}(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}) \qquad \sigma_{\theta\theta} = \frac{E}{(1-\nu^2)}(\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}) \qquad \tau_{r\theta} = G\gamma_{r\theta}$$

$\sigma_{rr} = \frac{-zE}{(1-\nu^2)}\left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)$	$\sigma_{rr} = \frac{-zE}{(1-\nu^2)}\left(\frac{d^2 w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)$
$\sigma_{\theta\theta} = \frac{-zE}{(1-\nu^2)}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \nu\frac{\partial^2 w}{\partial r^2}\right)$	$\sigma_{\theta\theta} = \frac{-zE}{(1-\nu^2)}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \nu\frac{d^2 w}{dr^2}\right)$
$\tau_{r\theta} = -2Gz\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial\theta}\right)$	$\tau_{r\theta} = 0$

4. Stress Resultants

$$\boldsymbol{m}_{rr} = \int_h \sigma_{rr}dz \qquad \boldsymbol{m}_{\theta\theta} = \int_h \sigma_{\theta\theta}dz \qquad \boldsymbol{m}_{r\theta} = \int_h \tau_{r\theta}dz \qquad \boldsymbol{m}_{\theta r} = \int_h \tau_{\theta r}dz \qquad \boldsymbol{q}_r = \int_h \tau_{rz}dz \qquad \boldsymbol{q}_\theta = \int_h \tau_{\theta z}dz$$



Internal Moments

$m_{rr} = -D\left(\dfrac{\partial^2 w}{\partial r^2} + \dfrac{\nu}{r}\dfrac{\partial w}{\partial r} + \dfrac{\nu}{r^2}\dfrac{\partial^2 w}{\partial \theta^2}\right)$	$m_{rr} = -D\left(\dfrac{d^2 w}{dr^2} + \dfrac{\nu}{r}\dfrac{\partial w}{\partial r}\right)$
$m_{\theta\theta} = -D\left(\dfrac{1}{r}\dfrac{\partial w}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial \theta^2} + \nu\dfrac{\partial^2 w}{\partial r^2}\right)$	$m_{\theta\theta} = -D\left(\dfrac{1}{r}\dfrac{dw}{dr} + \nu\dfrac{d^2 w}{dr^2}\right)$
$m_{r\theta} = -D(1-\nu)\left(\dfrac{1}{r}\dfrac{\partial^2 w}{\partial r \partial \theta} - \dfrac{1}{r^2}\dfrac{\partial w}{\partial \theta}\right)$	$m_{r\theta} = 0$

Internal Shear Forces

$q_r = -D\dfrac{\partial}{\partial r}(\nabla^2 w)$ $\quad q_\theta = -\dfrac{D}{r}\dfrac{\partial}{\partial \theta}(\nabla^2 w)$	$q_r = -D\dfrac{d}{dr}\left\{\dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dw}{dr}\right)\right\}$ $\quad q_\theta = 0$

Internal boundary shear forces

$V_r = q_r + \dfrac{1}{r}\dfrac{\partial m_{r\theta}}{\partial \theta}$ $\quad V_\theta = q_\theta + \dfrac{\partial m_{r\theta}}{\partial r}$	$V_r = q_r$ $\quad V_\theta = 0$

5. Equilibrium and Differential Equations

$$\nabla^2 \nabla^2 w = p_z(r, \theta)$$

$\nabla^2 = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left[r\dfrac{\partial}{\partial r}\right] + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$	$\nabla^2 = \dfrac{1}{r}\dfrac{d}{dr}\left[r\dfrac{d}{dr}\ \right]$

6. Stress Formulas

$\sigma_{rr} = \dfrac{\boldsymbol{m}_{rr} z}{(h^3/12)}$ $\sigma_{\theta\theta} = \dfrac{\boldsymbol{m}_{\theta\theta} z}{(h^3/12)}$ $\tau_{r\theta} = \dfrac{\boldsymbol{m}_{r\theta} z}{(h^3/12)}$	$\sigma_{rr} = \dfrac{\boldsymbol{m}_{rr} z}{(h^3/12)}$ $\sigma_{\theta\theta} = \dfrac{\boldsymbol{m}_{\theta\theta} z}{(h^3/12)}$ $\tau_{r\theta} = 0$

7. Boundary Conditions

$\boldsymbol{V}_r$ or w and $\boldsymbol{m}_{rr}$ or $\dfrac{\partial w}{\partial r}$	$\boldsymbol{q}_r$ or w and $\boldsymbol{m}_{rr}$ or $\dfrac{dw}{dr}$

Axisymmetric plate solution

$$\nabla^2\nabla^2 w = p_z(r,\theta) \qquad \nabla^2 = \frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\ \right] \text{ or}$$

$$\frac{D}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right] = p_z(r)$$

$$D\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right] = I_1(r) + C_1 \qquad \text{where} \qquad I_1(r) = \int r p_z(r) dr$$

$$D\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = I_2(r) + C_1 \ln r + C_2 \qquad \text{where} \qquad I_2(r) = \int \frac{I_1(r)}{r} dr$$

$$D\left(r\frac{dw}{dr}\right) = I_3(r) + C_1\left(\frac{r^2}{2}\ln r - \frac{r^2}{4}\right) + C_2\frac{r^2}{2} + C_3 \qquad \text{where} \qquad I_3(r) = \int r I_2(r) dr$$

$$Dw(r) = I_4(r) + \frac{C_1}{4}(r^2 \ln r - r^2) + C_2\frac{r^2}{4} + C_3 \ln r + C_4 \qquad \text{where} \qquad I_4(r) = \int \frac{I_3(r)}{r} dr$$

I's are the loading integrals.

Uniform Load $p_z(r) = p_o$

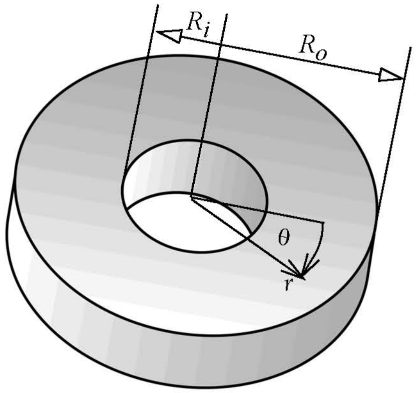
$$I_1(r) = p_o\int r dr = \frac{p_o r^2}{2} \qquad I_2(r) = \int \frac{I_1}{r} dr = \frac{p_o r^2}{4} \qquad I_3(r) = \int r I_2 dr = \frac{p_o r^4}{16} \qquad I_4(r) = \int \frac{I_3}{r} dr = \frac{p_o r^4}{64}$$

Solid Plate

- One boundary only.
- Displacement w and slope dw/dr must be bounded at $r = 0$

$$C_1 = 0 \qquad \text{and} \qquad C_3 = 0$$

Annular plates



Non-dimensional equations in annular plates

- The algebra for evaluating the constants in annular plates with distributed loads can be tedious because of the logarithmic function ($ln(r)$)
- Simplification in the algebra can be achieved by non-dimensionalizing the variables.

$$\hat{r} = r/R_i \qquad \hat{p}_o = p_z/p_o \qquad R = R_0/R_i \qquad 1 \le \hat{r} \le R$$

$$\hat{w} = \frac{Dw}{p_o R_i^4} \qquad \frac{d\hat{w}}{d\hat{r}} = \frac{D(dw/dr)}{p_o R_i^3} \qquad \hat{\boldsymbol{m}}_{rr} = \frac{\boldsymbol{m}_{rr}}{p_o R_i^2} \qquad \hat{\boldsymbol{m}}_{\theta\theta} = \frac{\boldsymbol{m}_{\theta\theta}}{p_o R_i^2} \qquad \hat{\boldsymbol{q}}_r = \frac{\boldsymbol{q}_r}{p_o R_i}$$

$$\hat{I}_1(\hat{r}) = \hat{r}^2/2 \qquad \hat{I}_2(\hat{r}) = \hat{r}^2/4 \qquad \hat{I}_3(\hat{r}) = \hat{r}^4/16 \qquad \hat{I}_4(\hat{r}) = \hat{r}^4/64$$

$$\hat{w}(\hat{r}) = \hat{I}_4(\hat{r}) + A_1(\hat{r}^2 ln\hat{r} - \hat{r}^2)/4 + A_2\hat{r}^2/4 + A_3 ln\hat{r} + A_4$$

$$\hat{r}\frac{d\hat{w}}{d\hat{r}} = \hat{I}_3(\hat{r}) + A_1\left(\frac{\hat{r}^2}{2}ln\hat{r} - \frac{\hat{r}^2}{4}\right) + A_2\frac{\hat{r}^2}{2} + A_3$$

$$\hat{\boldsymbol{m}}_{rr} = -[\hat{I}_2 - (\hat{I}_3/\hat{r}^2)(1-\nu)] - A_1[(1+\nu)ln\hat{r} + (1-\nu)/2]/2 - \frac{A_2(1+\nu)}{2} + A_3(1-\nu)/\hat{r}^2$$

$$\hat{\boldsymbol{m}}_{\theta\theta} = -[(\hat{I}_3/\hat{r}^2)(1-\nu) + \nu\hat{I}_2] - A_1[(1+\nu)ln\hat{r} - (1-\nu)/2]/2 - A_2(1+\nu)/2 - A_3(1-\nu)/\hat{r}^2$$

$$\hat{\boldsymbol{q}}_r = -(\hat{I}_1 + A_1)/\hat{r}$$

where, A_1, A_2, A_3, and A_4 are constants to be determine from boundary conditions.

- On inner boundary $\hat{r} = 1$ and $ln(\hat{r}) = 0$, simplifying algebra.

C5.4 An edge view of an annular plate that is attached to a rigid shaft, clamped at the outer edge and pulled with a force P as shown. Determine the maximum deflection w and maximum bending moments $\boldsymbol{m}_{rr}$ and $\boldsymbol{m}_{\theta\theta}$ in terms of D, P, and R_o. Use $\nu = 1/3$ and $R = 2$.