Plates

A plate is a flat solid body whose thickness is small compared to the other dimensions and is subjected to bending loads.

- Examples: Floors, ceilings, windows, disc brakes, ship decks, truck beds

The learning objectives

- Understand the theory of thin plate bending, its limitations, and its applications in design and analysis.
- Understand how to incorporate complexities in plate theory.

Limitations

1. The mid plane of the plate is flat.
2. The thickness of the plate $h(x,y)$ is an order of magnitude smaller than the dimensions in the $x$ or $y$ direction.
3. The thickness of the plate $h(x,y)$ and transverse loading $p_z(x,y)$ vary gradually.
4. We are away from the regions of stress concentration.
5. No in-plane loads -- pure bending.
6. $\max|\sigma_{zz}|$ and $\max\{|\sigma_{xx}|,|\sigma_{yy}|,|\tau_{xy}|\}$
7. The loads do not vary with time, or vary so slowly (quasi static problem)
8. We restrict our selves to linear theory.
   Kirchhoff-Love plate theory: $0 \leq w \leq 0.2h$ ---- Linear
   Von-Karman plate theory $0.3h \leq w \leq h$ ---- Non-linear theory.
   Membrane theory: $w \geq 2h$. 
Thin plate theory

- Our objective is to obtain equations relating the bending stresses $\sigma_{xx}$, $\sigma_{yy}$, and $\tau_{xy}$ to internal forces and moments and obtain the boundary value problem governing the transverse deflection $w$ of the plate

Assumption 1 Deformations are not functions of time.
Assumption 2 Squashing—dimensional changes in the $z$ direction, is significantly smaller than bending.

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \approx 0 \quad \Rightarrow \quad u = u(x, y, z) \quad v = v(x, y, z) \quad w = w(x, y)$$
Assumption 3 Plane sections before deformation remain planes after deformation.

- Displacements $u$ and $v$ are linear function of $z$.

\[
\frac{\partial w}{\partial x} \approx \frac{\partial w}{\partial x} \approx \frac{\partial w}{\partial x}
\]

\[
\tan^{-1}\left(\frac{\partial w}{\partial x}\right) \approx \left(\frac{\partial w}{\partial x}\right)
\]

\[
\tan^{-1}\left(\frac{\partial w}{\partial y}\right) \approx \left(\frac{\partial w}{\partial y}\right)
\]

\[
u \approx u_o(x, y) - z\sin\psi_y
\]

\[
v \approx v_o(x, y) - z\sin\psi_x
\]

\[
w \approx w(x, y)
\]

The rotations $\psi_x$ and $\psi_y$ are not functions $z$. $u_o(x, y) = 0$ \quad $v_o(x, y) = 0$ \quad If there are no inplane forces.
Assumption 4  Plane sections perpendicular to the plate mid surface remain nearly perpendicular after deformation.

The right angle at C remains a right angle at $C_F$. $\psi_y = \tan^{-1}\left(\frac{\partial w}{\partial x}\right)$ $\psi_x = \tan^{-1}\left(\frac{\partial w}{\partial y}\right)$

Assumption 5  Deflection and its derivatives are small.

$$sin\psi_x \approx \psi_x \quad and \quad sin\psi_y \approx \psi_y \quad tan^{-1}\left(\frac{\partial w}{\partial x}\right) \approx \left(\frac{\partial w}{\partial x}\right) \quad and \quad tan^{-1}\left(\frac{\partial w}{\partial y}\right) \approx \left(\frac{\partial w}{\partial y}\right)$$

$$u \approx -z\left(\frac{\partial w}{\partial x}\right) \quad v \approx -z\left(\frac{\partial w}{\partial y}\right) \quad w \approx w(x, y) \Rightarrow \gamma_{xx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \approx 0 \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \approx 0$$

Assumption 6  Strains are small

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

- The strains vary linearly with $z$.
- The maximum strain will be either on the top or bottom surface of the plate.
- Curvatures of deformed curved plate are $\partial^2 w/\partial x^2$, $\partial^2 w/\partial y^2$, and $\partial^2 w/\partial x \partial y$.

**Material Model**

Assumption 7  The material is isotropic.

Assumption 8  The material is linearly elastic.

Assumption 9  There are no thermal or non-mechanical strains.

- The plate is in state of **plane stress**. $\sigma_{xx} = \frac{E}{(1 - v^2)}(\varepsilon_{xx} + v\varepsilon_{yy})$ $\sigma_{yy} = \frac{E}{(1 - v^2)}(\varepsilon_{yy} + v\varepsilon_{xx})$ $\tau_{xy} = G\gamma_{xy}$

$$\sigma_{xx} = \frac{-Ez}{(1 - v^2)}\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right) \quad \sigma_{yy} = \frac{-Ez}{(1 - v^2)}\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right) \quad \tau_{xy} = -2Gz\frac{\partial^2 w}{\partial x \partial y}$$
Static Equivalency

\[
\begin{align*}
n_{xx} &= \int_{h}^{\infty} \sigma_{xx} \, dz \\
n_{yy} &= \int_{h}^{\infty} \sigma_{yy} \, dz \\
n_{xy} &= \int_{h}^{\infty} \sigma_{xy} \, dz \\
m_{xx} &= \int_{h}^{\infty} z \sigma_{xx} \, dz \\
m_{yy} &= \int_{h}^{\infty} z \sigma_{yy} \, dz \\
m_{xy} &= \int_{h}^{\infty} z \tau_{xy} \, dz \\
qu_x &= \int_{h}^{\infty} \tau_{xz} \, dz \\
qu_y &= \int_{h}^{\infty} \tau_{yz} \, dz
\end{align*}
\]

- All force stress resultants will have the units of force per unit length.
- All moment stress resultants will have the units of moments per unit length.
- The internal shear forces \(q_x\) and \(q_y\) are necessary for force equilibrium in the \(z\) direction, which implies that the stresses \(\tau_{xz}\) and \(\tau_{yz}\) cannot be zero but must be small.
- Average values of these stresses can be found by dividing the internal shear forces \(q_x\) and \(q_y\) by the thickness.
- The maximum of these average values must be an order of magnitude smaller than the maximum inplane stresses.

\[
\text{max} |\tau_{xz}| \text{ and } \text{max} |\tau_{yz}| \ll \text{max} \{ |\sigma_{xx}|, |\sigma_{yy}|, |\tau_{xy}| \}
\]
Location of neutral surface (origin)

\[ n_{xx} = \frac{\partial^2 w}{\partial x^2} \int_h^z \frac{Ez}{(1 - v^2)} \, dz + \frac{\partial^2 w}{\partial x \partial y} \int_h^z \frac{Evz}{(1 - v^2)} \, dz = 0 \]
\[ n_{yy} = \frac{\partial^2 w}{\partial y^2} \int_h^z \frac{Ez}{(1 - v^2)} \, dz + \frac{\partial^2 w}{\partial x \partial y} \int_h^z \frac{Evz}{(1 - v^2)} \, dz = 0 \]
\[ n_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y} \int_h^z Gzdz = 0 \]

Assumption 10: The material is homogeneous across the thickness of the plate.

\[ n_{xx} = \left[ \frac{\partial^2 w}{\partial x^2} \frac{E}{(1 - v^2)} + \frac{\partial^2 w}{\partial y^2} \frac{Ev}{(1 - v^2)} \right] \int_h^z \, dz = 0 \]
\[ n_{xy} = \left[ \frac{\partial^2 w}{\partial x \partial y} \frac{E}{(1 - v^2)} + \frac{\partial^2 w}{\partial x^2} \frac{Ev}{(1 - v^2)} \right] \int_h^z \, dz = 0 \]
\[ n_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y} \int_h^z \, dz \]

- \[ \int_h^z \, dz = 0 \]: The origin and the neutral surface must be at the mid surface for pure bending of plates.

Stress formulas

\[ m_{xx} = -\left( \frac{\partial^2 w}{\partial x^2} \right) \int_h^z \frac{Ez^2}{(1 - v^2)} \, dz - \left( \frac{\partial^2 w}{\partial y^2} \right) \int_h^z \frac{Evz^2}{(1 - v^2)} \, dz \]
\[ m_{yy} = -\left( \frac{\partial^2 w}{\partial y^2} \right) \int_h^z \frac{Ez^2}{(1 - v^2)} \, dz - \left( \frac{\partial^2 w}{\partial x^2} \right) \int_h^z \frac{Evz^2}{(1 - v^2)} \, dz \]
\[ m_{xy} = -2 \left( \frac{\partial^2 w}{\partial x \partial y} \right) \int_h^z Gz^2 \, dz \]

As per Assumption 10 of homogeneity across the thickness, we obtain

\[ m_{xx} = -\left[ \frac{E}{(1 - v^2)} \right] \left[ \left( \frac{\partial^2 w}{\partial x^2} \right) \int_h^z z^2 \, dz + \left( \frac{\partial^2 w}{\partial y^2} \right) \int_h^z z^2 \, dz \right] \]
\[ m_{yy} = -\left[ \frac{E}{(1 - v^2)} \right] \left[ \left( \frac{\partial^2 w}{\partial y^2} \right) \int_h^z z^2 \, dz + \left( \frac{\partial^2 w}{\partial x^2} \right) \int_h^z z^2 \, dz \right] \]
\[ m_{xy} = -2G \left( \frac{\partial^2 w}{\partial x \partial y} \right) \int_h^z z^2 \, dz \]

\[ \int_h^z z^2 \, dz = \int_{-h/2}^{h/2} z^2 \, dz = h^3 / 12 \]

Moment-curvature equations

\[ m_{xx} = -D \left[ \left( \frac{\partial^2 w}{\partial x^2} \right) + v \left( \frac{\partial^2 w}{\partial y^2} \right) \right] \]
\[ m_{yy} = -D \left[ \left( \frac{\partial^2 w}{\partial y^2} \right) + v \left( \frac{\partial^2 w}{\partial x^2} \right) \right] \]
\[ m_{xy} = -D(1 - v) \left( \frac{\partial^2 w}{\partial x \partial y} \right) \]

\[ D = \frac{Eh^3}{12(1 - v^2)} \]

- \[ D \]: bending rigidity of the plate.

\[ \sigma_{xx} = \frac{m_{xx}}{(h^3/12)} \]
\[ \sigma_{yy} = \frac{m_{yy}}{(h^3/12)} \]
\[ \tau_{xy} = \frac{m_{xy}}{(h^3/12)} \]
• The bending stresses in plates vary linearly through the thickness and will be maximum at top and bottom surface of the plate.

**Equilibrium**

Force equilibrium in $z$-direction:

\[
\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = -p_z(x, y)
\]

![Diagram showing force equilibrium and moment equilibrium](image)

Equilibrium of moment about the $x$ and $y$ direction at point O.

\[
q_x = \frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{yx}}{\partial y} \quad q_y = \frac{\partial m_{xy}}{\partial x} + \frac{\partial m_{yy}}{\partial y}
\]
**Differential equation**

\[ q_x = -\frac{\partial}{\partial x} \left[ D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \right] - \frac{\partial}{\partial y} \left[ D(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \right] \]

\[ q_y = -\frac{\partial}{\partial x} \left[ \left( D(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \right) \right] - \frac{\partial}{\partial y} \left[ D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \right] \]

\[ \frac{\partial^2}{\partial x^2} \left[ D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \right] + 2\frac{\partial^2}{\partial x \partial y} \left[ D(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \right] + \frac{\partial^2}{\partial y^2} \left[ D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \right] = p_z(x,y) \]

**Assumption 11** The plate is homogeneous in the \( x \) and \( y \) direction.

**Assumption 12** The plate is of uniform thickness.

\[ h, E, \nu, \text{ and } D \text{ are constant.} \]

\[ q_x = -D \frac{\partial}{\partial x} \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] = -D \frac{\partial}{\partial x} (\nabla^2 w) \quad \text{and} \quad q_y = -D \frac{\partial}{\partial y} \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] = -D \frac{\partial}{\partial y} (\nabla^2 w) \]

\[ D \nabla^4 w = D \left[ \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] = p_z(x,y) \]

where, \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the Laplace operator and \( \nabla^4 = \nabla^2 \nabla^2 \) is the bi-harmonic operator.
Boundary Conditions
Fourth order partial differential equation requires two conditions at each boundary point. On boundary point there are three internal quantities, a shear force and two moments. These three quantities need to be reduced to two.

Kirchhoff's arguments

\[ V_x = q_x + \frac{\partial m_{xy}}{\partial y} = -D \left[ \frac{\partial^3 w}{\partial x^3} + (2 - \nu) \frac{\partial^3 w}{\partial x \partial y^2} \right] \]

\[ V_y = q_y + \frac{\partial m_{yx}}{\partial x} = -D \left[ \frac{\partial^3 w}{\partial y^3} + (2 - \nu) \frac{\partial^3 w}{\partial y \partial x^2} \right] \]

Corner Force

\[ R_{\text{corner}} = m_{xy} + m_{yx} = 2m_{xy} = -2D(1 - \nu) \frac{\partial^2 w}{\partial x \partial y} \]

On \( x = \text{constant} \) Specify \([V_x \text{ or } w]\) and \([m_{xx} \text{ or } \frac{\partial^3 w}{\partial x^2}]\)

On \( y = \text{constant} \) Specify \([V_y \text{ or } w]\) and \([m_{yy} \text{ or } \frac{\partial^3 w}{\partial y^2}]\)
<table>
<thead>
<tr>
<th>Type of support</th>
<th>Boundary Condition</th>
<th>Specify ( V_x ) or ( w )</th>
<th>Specify ( m_{xx} ) or ( \frac{\partial w}{\partial x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clamped</td>
<td>( w(a, y) = 0 )</td>
<td>( \frac{\partial w}{\partial x} (a, y) = 0 )</td>
<td></td>
</tr>
<tr>
<td>Simply Supported</td>
<td>( w(a, y) = 0 )</td>
<td>( m_{xx}(a, y) = 0 ) or ( \frac{\partial^2 w}{\partial y^2}(a, y) = 0 )</td>
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<tr>
<td>Free</td>
<td>( V_x(a, y) = 0 ) or ( \left[ \frac{\partial^3 w}{\partial y^3} + (2 - v) \frac{\partial^3 w}{\partial y \partial x^2} \right] \bigg</td>
<td>_{x = a} = 0 )</td>
<td>( m_{xx}(a, y) = 0 ) or ( \left[ \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right] \bigg</td>
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<tr>
<td>Roller</td>
<td>( V_x(a, y) = 0 ) or ( \left[ \frac{\partial^3 w}{\partial y^3} + (2 - v) \frac{\partial^3 w}{\partial y \partial x^2} \right] \bigg</td>
<td>_{x = a} = 0 )</td>
<td>( \frac{\partial w}{\partial x} (a, y) = 0 )</td>
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<tr>
<td>Elastic</td>
<td>( V_x(a, y) = 0 = \frac{}{}K_L w(a, y) )</td>
<td>( m_{xx}(a, y) = \frac{}{}K_\theta \frac{\partial w}{\partial x} (a, y) )</td>
<td></td>
</tr>
</tbody>
</table>

\[ m_{xx}(a, y) \quad \leftarrow \quad \frac{}{}x = a \quad \rightarrow \quad \frac{}{}K_\theta \frac{\partial w}{\partial x} (a, y) \]

\[ V_x(a, y) \quad \leftarrow \quad \frac{}{}K_L w(a, y) \]
C5.1 The cross section of laminated plate made from two materials is shown below. Both materials have the same Poisson’s ratio but different modulus of elasticity. The displacement field is given

\[ u \approx -(z - z_o)(\frac{\partial w}{\partial x}) \quad v \approx -(z - z_o)(\frac{\partial w}{\partial y}) \quad w \approx w(x, y) \]

where, \( z \) is measured from the mid surface and \( z_o \) is the location of the neutral surface. 

(a) Determine the value of \( z_o \) assuming all assumptions except for material homogeneity across the thickness are valid. 

(b) Obtain stress formulas and the differential equation governing the deflection of the laminated plate. 

(c) By substituting \( E_1 = E_2 \), show that the results for parts a and b give the same results as classical plate theory for homogeneous material.
C5.2 In *Mindlin-Reissner* plate theory Assumption 4 of planes sections perpendicular to the plate mid surface remain nearly perpendicular after deformation is dropped to account for shear. (a) Starting with the displacement field below obtain the stress formulas and differential equations if all assumptions except Assumption 4 are valid.

\[ u \approx -z \psi_x(x, y) \quad v \approx -z \psi_y(x, y) \quad w \approx w(x, y) \]

(b) Show the results reduce to those of classical plate theory by substituting \( \psi_x = \frac{\partial w}{\partial x} \) and \( \psi_y = \frac{\partial w}{\partial y} \).
Navier’s solution of rectangular plates

In 1820, Navier found the solution for a rectangular plate that is simply supported on all its boundary.

Differential Equation

\[
\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{P_z(x,y)}{D}
\]

Boundary Conditions

On \( x = 0 \) \( w(0,y) = 0 \) \( \frac{\partial^2 w}{\partial x^2}(0,y) = 0 \) On \( x = a \) \( w(a,y) = 0 \) \( \frac{\partial^2 w}{\partial x^2}(a,y) = 0 \)

On \( y = 0 \) \( w(x,0) = 0 \) \( \frac{\partial^2 w}{\partial y^2}(x,0) = 0 \) On \( y = b \) \( w(x,b) = 0 \) \( \frac{\partial^2 w}{\partial y^2}(x,b) = 0 \)

- Only even derivatives in the boundary value problem.
- Even derivatives of Sine (Cosines) functions produce Sine (Cosine) functions.
- Sine functions can satisfy all boundary conditions.
\[
    w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \\
p_z(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)
\]

where, \( W_{mn} \) and \( P_{mn} \) are constant coefficients to be determined.

- The series consists of independent functions.
- The infinite series is complete for this class of problem.

Substituting \( w(x, y) \) and \( p_z(x, y) \) we obtain

\[
    W_{mn} = \frac{1}{D\pi^4} \frac{P_{mn}}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}
\]

To determine the constants \( P_{mn} \) we use the orthogonality conditions of Sine functions given below.

**Orthogonality condition:**

\[
    \int_{0}^{\pi} \sin(m\theta) \sin(n\theta) \, d\theta = \begin{cases} 
        0 & m \neq n \\
        \frac{\pi}{2} & m = n
    \end{cases}
\]

\[
    P_{mn} = \frac{4}{ab} \int_{0}^{a} \int_{0}^{b} p_z(x, y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \, dx \, dy
\]
**Uniform distributed load** $p_0(x, y) = p_o$

$$P_{mn} = \begin{cases} 
0 & m, n = 2, 4, 6, \ldots \\
\frac{16p_o}{\pi^2 mn} & m, n = 1, 3, 5, \ldots
\end{cases}$$

$$W_{mn} = \begin{cases} 
0 & m, n = 2, 4, 6, \ldots \\
\frac{16p_o}{D\pi^6 mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]} & m, n = 1, 3, 5, \ldots
\end{cases}$$

$$w(x, y) = \frac{16p_o}{D\pi^6} \sum_{m=1, 3, 5, \ldots}^{\infty} \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{1}{mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]} \sin\left(m\pi\frac{x}{a}\right) \sin\left(n\pi\frac{y}{b}\right)$$

$$m_{xx}(x, y) = \left(\frac{16p_o}{\pi^4} \right) \sum_{m=1, 3, 5, \ldots}^{\infty} \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}{mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2} \sin\left(m\pi\frac{x}{a}\right) \sin\left(n\pi\frac{y}{b}\right)$$

Maximum values of $w$ and $m_{xx}$ at $x = a/2$ and $y = b/2$ ----Center of plate

**Convergence Study**

Square plate: $b = a$  Poisson’s ratio of $\nu = 1/3$.

$$w_{max} = \frac{16p_o a^4}{D\pi^6} \sum_{m=1, 3, 5, \ldots}^{\infty} \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{(-1)^{(m+n-2)/2}}{mn\left[m^2 + n^2\right]^2}$$

$$(m_{xx})_{max} = \left(\frac{16p_o a^2}{\pi^4} \right) \sum_{m=1, 3, 5, \ldots}^{\infty} \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{\left[m^2 + \nu n^2\right](-1)^{(m+n-2)/2}}{mn\left[m^2 + n^2\right]^2}$$
\[
\text{% difference} = \frac{T_n - T_{16}}{T_{16}} \times 100 \text{ where } T_n \text{ is the value after } n \text{ terms.}
\]

### Convergence Study

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<th>No. of terms</th>
<th>m</th>
<th>n</th>
<th>(\frac{w_{\text{max}}(10^{-3})}{(p_0a^4)/D}) value</th>
<th>(\text{% difference})</th>
<th>(\frac{(m_{xx})_{\text{max}}(10^{-3})}{p_0a^2}) value</th>
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\[ \frac{w_{\text{max}}(10^{-3})}{(p_0a^4)/D} \text{ vs. number of terms} \]

\[ \frac{(m_{xx})_{\text{max}}(10^{-3})}{p_0a^2} \text{ vs. number of terms} \]

- Differentiation increases and integration decreases the error of approximation.
Nadai-Levy solution of rectangular plates

1. Plate is simply supported on two opposite sides as shown below. The other two sides can have any type of support or boundary conditions. The boundary conditions at $x = 0$ and $x = a$ can be written as

$$w(0, y) = 0 \quad \frac{\partial^2 w}{\partial x^2}(0, y) = 0 \quad w(a, y) = 0 \quad \frac{\partial^2 w}{\partial x^2}(a, y) = 0$$

(5.1a)

2. The loading $p_z$ is only dependent on $x$, that is, $p_z(x, y) = p_z(x)$.

- the deflection solution in two parts: $w(x, y) = w_p(x) + w_h(x, y)$

$$\frac{d^4 w_p}{dx^4} = \frac{p_z(x)}{D} \quad \frac{\partial^4 w_h}{\partial x^4} + 2 \frac{\partial^4 w_h}{\partial x^2 \partial y^2} + \frac{\partial^4 w_h}{\partial y^4} = 0$$

(5.2a)
Homogeneous solution

\[ w_h(x, y) = \sum_{m=1}^{\infty} H_m(y) \sin(m \pi x / a) \]  

Substituting

\[ \left( \frac{m \pi}{a} \right)^4 H_m - 2 \left( \frac{m \pi}{a} \right)^2 \frac{d^2 H_m}{dy^2} + \frac{d^4 H_m}{dy^4} = 0 \]

The homogeneous solution is

\[ w_h(x, y) = \sum_{m=1}^{\infty} \left\{ [A_m + D_m(m \pi y / a)] \cosh(m \pi y / a) + [C_m + B_m(m \pi y / a)] \sinh(m \pi y / a) \right\} \sin(m \pi y / a) \]

Particular solution

Method I:

\[ w_p(x) = \sum_{m=1}^{\infty} W_m \sin(m \pi x / a) \]

\[ p_z(x) = \sum_{m=1}^{\infty} P_m \sin(m \pi x / a) \]

\[ W_m = \left( \frac{a}{mn} \right)^4 \frac{P_m}{D} \]

\[ P_m = \frac{2}{a} \int_0^a p_z(x) \sin(m \pi x / a) dx \]

Method II: Direct integration for given value of \( p_z \)

For uniform load \( p_z = p_0 \).

\[ w_p = \frac{p_0 a^4}{24D} \left[ \left( \frac{x}{a} \right)^4 - 2 \left( \frac{x}{a} \right)^3 + \left( \frac{x}{a} \right) \right] \]
C5.3 The simply supported plate shown below is subject to a uniform distributed force of $p_o$. (a) Obtain a series solution for deflection $w$ using Nadai-Levy Method. (b) For maximum deflection $w$ and bending moment $m_{xx}$, compare the convergence of Nadai-Levy Method with Navier’s method for a square plate and Poisson’s ratio of $\nu = 1/3$. 

![Diagram of a simply supported plate with uniform distributed force $p_o$. The plate has dimensions $a$ and $b$ and is simply supported along the edges.](image)
Circular plates

Axisymmetric: Loading, material property, and geometry are independent of the angular coordinate \( \theta \).

Solution is independent of angular coordinate \( \theta \)

\[
\begin{align*}
tw &= w(r) \\
\partial w / \partial \theta &= 0 \\
\partial w / \partial r &= dw / dr
\end{align*}
\]

1. Displacements

\[
\begin{align*}
\varepsilon_{rr} &= z \left( \frac{\partial^2 w}{\partial r^2} \right) \\
v_\theta &= z \left( \frac{\partial w}{\partial \theta} \right) \\
w &\approx w(r) \\
\end{align*}
\]

\[
\begin{align*}
u_r &= -\frac{dw}{dr} \\
v_\theta &= 0 \\
w &\approx w(r)
\end{align*}
\]

2. Strains

\[
\begin{align*}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} = \frac{1}{r} \left( \frac{\partial w}{\partial \theta} \right) \\
\varepsilon_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} = z \left( \frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right) \\
\gamma_{r\theta} &= \frac{1}{r} \frac{\partial u_r}{\partial \theta} = - \frac{2}{r} z \left( \frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right)
\end{align*}
\]
3. Stresses

\[ \sigma_{rr} = \frac{E}{(1 - \nu^2)} (\varepsilon_{rr} + \nu \varepsilon_{\theta\theta}) \quad \sigma_{\theta\theta} = \frac{E}{(1 - \nu^2)} (\varepsilon_{\theta\theta} + \nu \varepsilon_{rr}) \quad \tau_{r\theta} = G \gamma_{r\theta} \]

\[ \sigma_{rr} = -z E \left( \frac{\partial^2 w}{\partial r^2} + \frac{\nu \partial w}{r \partial r} + \frac{\nu \partial^2 w}{r^2 \partial \theta^2} \right) \quad \sigma_{\theta\theta} = -z E \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{\nu \partial^2 w}{r \partial r} \right) \quad \tau_{r\theta} = 0 \]

4. Stress Resultants

\[ m_{rr} = \int_{h} \sigma_{rr} dz \quad m_{\theta\theta} = \int_{h} \sigma_{\theta\theta} dz \quad m_{r\theta} = \int_{h} \tau_{r\theta} dz \quad m_{0r} = \int_{h} \tau_{\theta\theta} dz \quad q_r = \int_{h} \tau_{rr} dz \quad q_\theta = \int_{h} \tau_{r\theta} dz \]
Internal Moments

\[ m_{rr} = -D \left( \frac{\partial^2 w}{\partial r^2} + \frac{\nu \partial w}{r \partial r} + \frac{\nu}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \]
\[ m_{\theta \theta} = -D \left( \frac{\partial \partial w}{\partial r \partial \theta} + \frac{\partial^2 w}{r \partial r^2} + \frac{\partial^2 w}{\partial r \partial \theta} \right) \]
\[ m_{r \theta} = -D (1 - \nu) \left( \frac{\partial w}{\partial r} - \frac{\partial w}{\partial \theta} \right) \]

Internal Shear Forces

\[ q_r = -D \frac{\partial}{\partial r} (\nabla^2 w) \]
\[ q_\theta = -D \frac{\partial}{\partial \theta} (\nabla^2 w) \]
\[ q_r = -D \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} \left( r \frac{dw}{dr} \right) \right) \]
\[ q_\theta = 0 \]

Internal boundary shear forces

\[ V_r = q_r + \frac{\partial m_{r \theta}}{\partial \theta} \]
\[ V_\theta = q_\theta + \frac{\partial m_{r \theta}}{\partial r} \]
\[ V_r = q_r \]
\[ V_\theta = 0 \]

5. Equilibrium and Differential Equations

\[ \nabla^2 \nabla^2 w = p_z (r, \theta) \]

\[ \nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \]
\[ \nabla^2 = \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) \]
6. Stress Formulas

\[
\sigma_{rr} = \frac{m_{rr}z}{(h^3/12)} \\
\sigma_{\theta\theta} = \frac{m_{\theta\theta}z}{(h^3/12)} \\
\tau_{r\theta} = \frac{m_{r\theta}z}{(h^3/12)}
\]

\[
\sigma_{rr} = \frac{m_{rr}z}{(h^3/12)} \\
\sigma_{\theta\theta} = \frac{m_{\theta\theta}z}{(h^3/12)} \\
\tau_{r\theta} = 0
\]

7. Boundary Conditions

<table>
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<th>(V_r) or (w) and</th>
<th>(q_r) or (w) and</th>
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<tr>
<td>(m_{rr}) or (\frac{\partial w}{\partial r})</td>
<td>(m_{rr}) or (\frac{\partial w}{\partial r})</td>
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Axisymmetric plate solution

\[ \nabla^2 \nabla^2 w = p_z(r, \theta) \quad \nabla^2 = \frac{1}{r \frac{d}{dr}} \left[ r \frac{d}{dr} \right] \text{ or} \]

\[ \frac{D}{r \frac{d}{dr}} \left[ r \frac{d}{dr} \left( \frac{1}{r \frac{d}{dr}} \frac{d w}{d r} \right) \right] = p_z(r) \]

\[ D \left[ \frac{d}{dr} \left( \frac{1}{r \frac{d}{dr}} \frac{d w}{d r} \right) \right] = I_1(r) + C_1 \quad \text{where} \quad I_1(r) = \int r p_z(r) dr \]

\[ D \left( \frac{d w}{d r} \right) = r I_2(r) + C_1 \ln r + C_2 \quad \text{where} \quad I_2(r) = \int \frac{I_1(r)}{r} dr \]

\[ D \left( \frac{d w}{d r} \right) = r I_3(r) + C_1 \left( \frac{r^2}{2} \ln r - \frac{r^2}{4} \right) + C_2 \frac{r^2}{2} + C_3 \quad \text{where} \quad I_3(r) = \int r I_2(r) dr \]

\[ D w(r) = I_4(r) + C_1 \frac{r^2}{4} (r^2 \ln r - r^2) + C_2 \frac{r^2}{4} + C_3 \ln r + C_4 \quad \text{where} \quad I_4(r) = \int \frac{I_3(r)}{r} dr \]

I’s are the loading integrals.

**Uniform Load** \( p_z(r) = p_o \)

\[ I_1(r) = p_o \int r dr = \frac{p_o r^2}{2} \quad I_2(r) = \int \frac{I_1}{r} dr = \frac{p_o r^2}{4} \quad I_3(r) = \int r I_2 dr = \frac{p_o r^4}{16} \quad I_4(r) = \int \frac{I_3}{r} dr = \frac{p_o r^4}{64} \]

**Solid Plate**

- One boundary only.
- Displacement \( w \) and slope \( \frac{d w}{d r} \) must be bounded at \( r = 0 \)

\[ C_1 = 0 \quad \text{and} \quad C_3 = 0 \]
Annular plates

Non-dimensional equations in annular plates

- The algebra for evaluating the constants in annular plates with distributed loads can be tedious because of the logarithmic function ($ln(r)$)
- Simplification in the algebra can be achieved by non-dimensionalizing the variables.

$$\bar{r} = r/R_i \quad \bar{p}_o = p_o/P_o \quad R = R_0/R_i \quad 1 \leq \bar{r} \leq R$$

$$\bar{w} = \frac{Dw}{p_o R_i^4} \quad \frac{d\bar{w}}{d\bar{r}} = \frac{D(dw/dr)}{p_o R_i^3} \quad \bar{m}_{rr} = \frac{m_{rr}}{p_o R_i^2} \quad \bar{m}_{\theta\theta} = \frac{m_{\theta\theta}}{p_o R_i^2} \quad \bar{q}_r = \frac{q_r}{p_o R_i}$$

$$\bar{I}_1(\bar{r}) = \frac{\bar{r}^2}{2} \quad \bar{I}_2(\bar{r}) = \frac{\bar{r}^2}{4} \quad \bar{I}_3(\bar{r}) = \frac{\bar{r}^4}{16} \quad \bar{I}_4(\bar{r}) = \frac{\bar{r}^4}{64}$$

$$\bar{w}(\bar{r}) = \bar{I}_4(\bar{r}) + A_1(\bar{r}^2 ln \bar{r} - \bar{r}^2)/4 + A_2(\bar{r}^2)/4 + A_3 ln \bar{r} + A_4$$

$$\bar{r} \frac{d\bar{w}}{d\bar{r}} = \bar{I}_3(\bar{r}) + A_1(\bar{r}^2 ln \bar{r} - \bar{r}^2)/4 + A_2\frac{\bar{r}^2}{2} + A_3$$

$$\bar{m}_{rr} = -[\bar{I}_2 - (\bar{I}_3 / \bar{r}^2)(1-v)] - A_1[(1 + v) ln \bar{r} + (1 - v)/2]/2 - \frac{A_2(1 + v)}{2} + A_3(1 - v)/\bar{r}^2$$

$$\bar{m}_{\theta\theta} = -[(\bar{I}_3 / \bar{r}^2)(1-v) + v\bar{I}_2] - A_1[(1 + v) ln \bar{r} - (1 - v)/2]/2 - A_2(1 + v)/2 - A_3(1 - v)/\bar{r}^2$$

$$\bar{q}_r = - (\bar{I}_1 + A_1)/\bar{r}$$

where, $A_1, A_2, A_3, \text{ and } A_4$ are constants to be determined from boundary conditions.

- On inner boundary $\bar{r} = 1$ and $ln(\bar{r}) = 0$, simplifying algebra.
C5.4 An edge view of an annular plate that is attached to a rigid shaft, clamped at the outer edge and pulled with a force $P$ as shown. Determine the maximum deflection $w$ and maximum bending moments $m_{rr}$ and $m_{\theta\theta}$ in terms of $D$, $P$, and $R_o$. Use $\nu = 1/3$ and $R = 2$. 