Stress and Strain

(a) Inadequate Strength

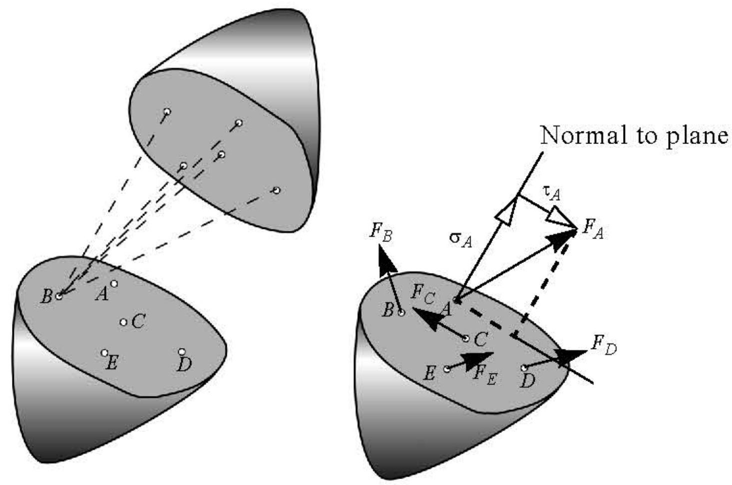
(b) Adequate Stiffness

Courtesy (a) *Rijan*. http://en.wikipedia.org/wiki/File:Dhaka_Savar_Building_Collapse.jpg (b) *Mate 2nd Class Isaiah Sellers III*. http://commons.wikimedia.org/wiki/File:Diving.jpg

The learning objectives in this chapter are:

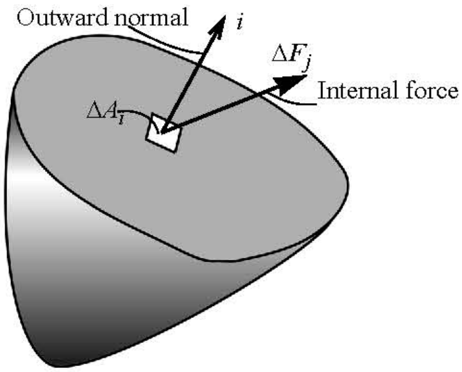
- To understand the concepts of stress and strain.
- To understand stress and strain transformations in three dimensions.
- To understand the relationship of stress to internal forces and moments.

Internally Distributed Force System



- The intensity of internal distributed forces on an imaginary cut surface of a body is called the *stress on a surface*.
- The intensity of internal distributed force that is normal to the surface of an imaginary cut is called the *normal stress* on a surface.
- The intensity of internal distributed force that is parallel to the surface of an imaginary cut surface is called the *shear stress* on the surface.

Stress at a Point



$$\sigma_{ij} = \lim_{\Delta A_i \to 0}\left(\frac{\Delta F_j}{\Delta A_i}\right)$$

direction of outward normal to the imaginary cut surface.

direction of the internal force component.

- ΔA_i will be considered positive if the outward normal to the surface is in the positive i direction.
- A stress component is positive if numerator and denominator have the same sign. Thus σ_{ij} is positive if: (1) ΔF_j and ΔA_i are both positive. (2) ΔF_j and ΔA_i are both negative.

- Stress Matrix in 3-D: $\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$

Table 1.1. Comparison of number of components

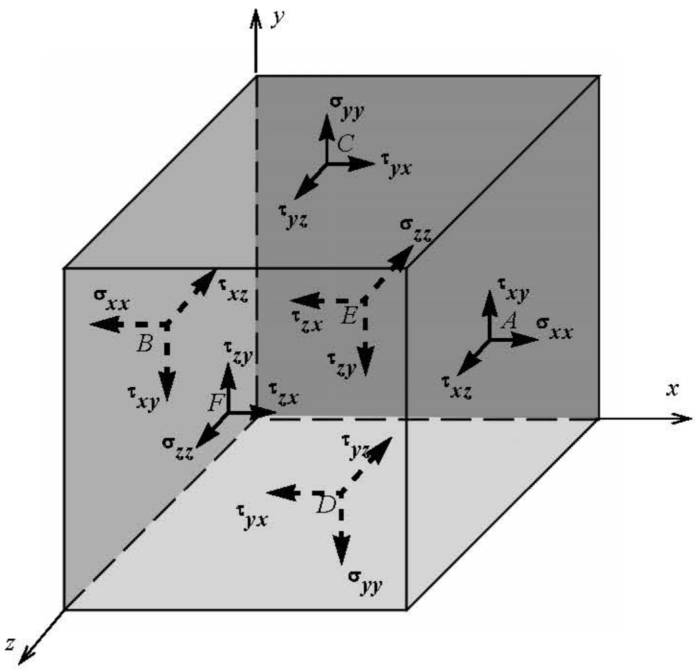
Quantity	1-D	2-D	3-D
Scaler	$1=1^0$	$1=2^0$	$1=3^0$
Vector	$1=1^1$	$2=2^1$	$3=3^1$
Stress	$1=1^2$	$4=2^2$	$9=3^2$

Stress Element

- Stress element is an imaginary object that helps us visualize stress at a point by constructing surfaces that have outward normal in the coordinate directions.

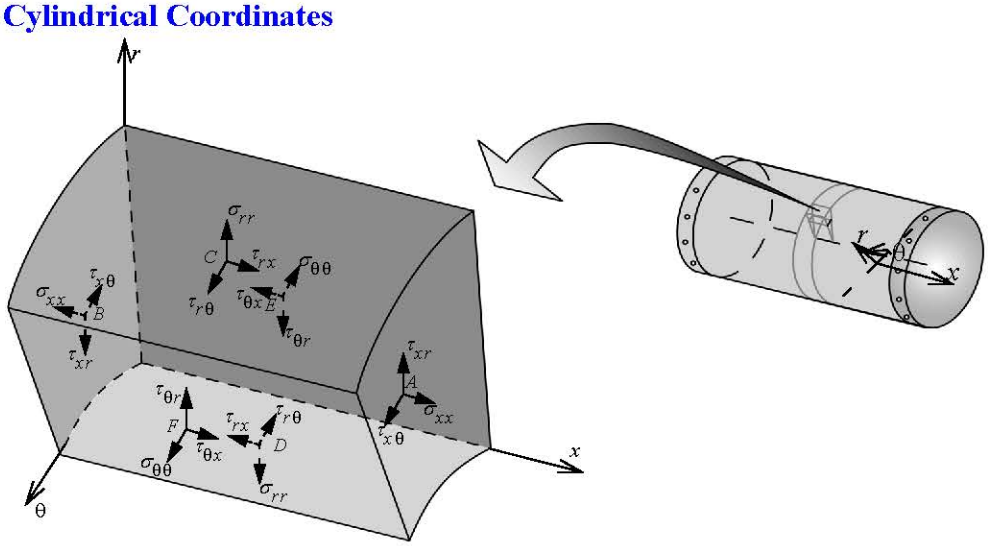
Stress Element in Cartesian Coordinates

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$



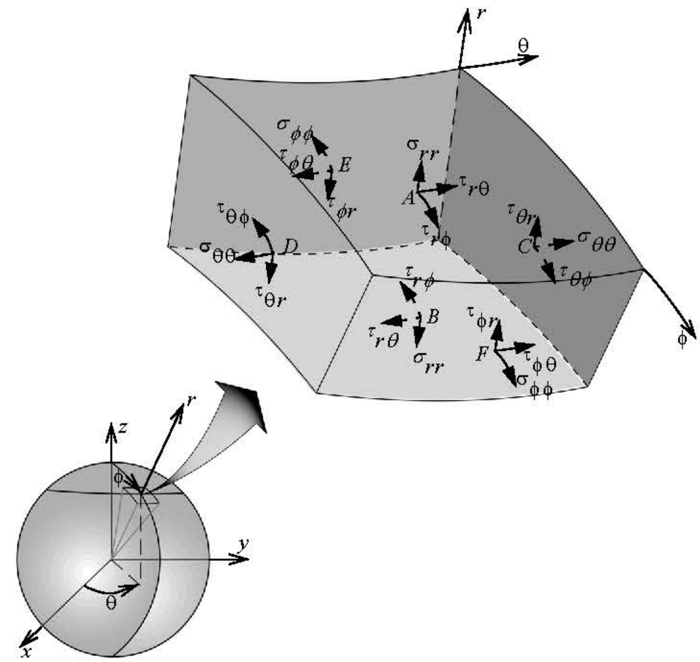
Stress Element in Cylindrical Coordinates

$$\begin{bmatrix} \sigma_{xx} & \tau_{xr} & \tau_{x\theta} \\ \tau_{rx} & \sigma_{rr} & \tau_{r\theta} \\ \tau_{\theta x} & \tau_{\theta r} & \sigma_{\theta\theta} \end{bmatrix}$$



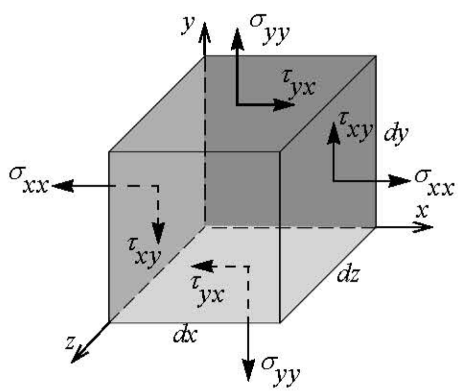
Stress Element in Spherical Coordinates

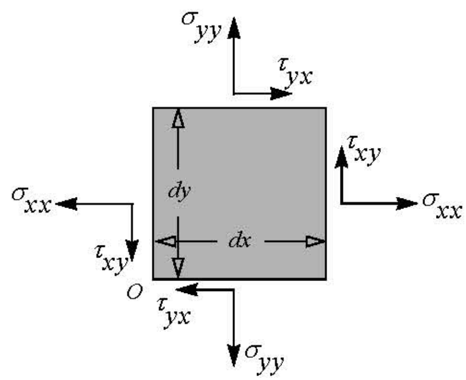
$$\begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{r\phi} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta\phi} \\ \tau_{\phi r} & \tau_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix}$$



Plane Stress: All stress components on a plane are zero.

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$





Symmetric Shear Stresses: $\tau_{xy} = \tau_{yx}$ $\tau_{yz} = \tau_{zy}$ $\tau_{zx} = \tau_{xz}$

- A pair of symmetric shear stress points towards the corner or away from the corner.

C1.1 Show the non-zero stress components on the A,B, and C faces of the cube shown below.

$$\begin{bmatrix} \sigma_{xx} = 0 & \tau_{xy} = -15ksi & \tau_{xz} = 0 \\ \tau_{yx} = -15ksi & \sigma_{yy} = 10ksi(C) & \tau_{yz} = 25ksi \\ \tau_{zx} = 0 & \tau_{zy} = 25ksi & \sigma_{zz} = 20ksi(T) \end{bmatrix}$$

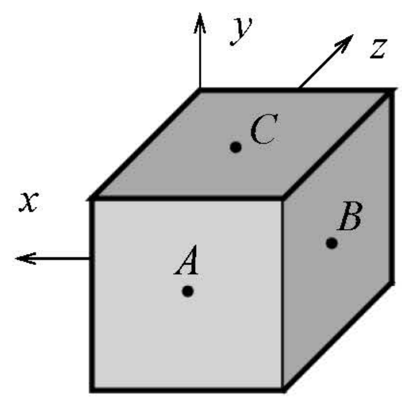
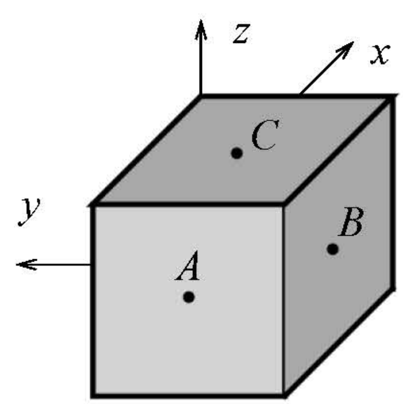


Fig. P1.1

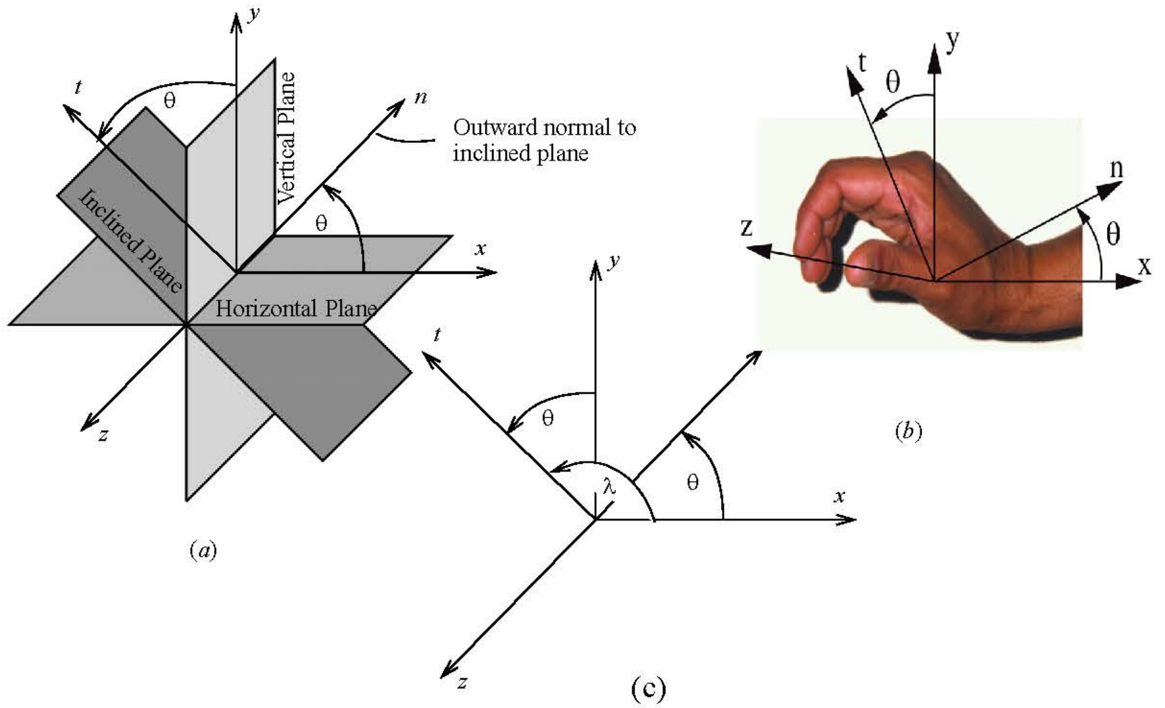
Class Problem 1.1

Show the non-zero stress components on the A,B, and C faces of the cube shown below.

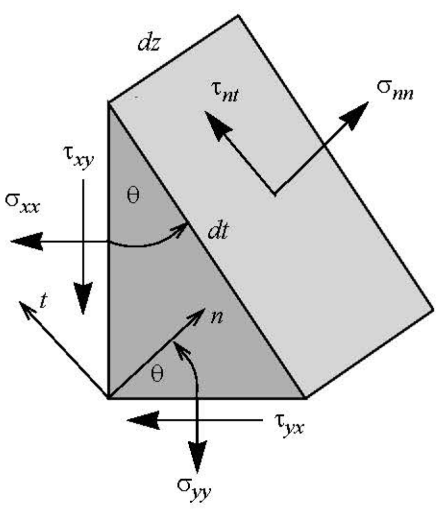
$$\begin{bmatrix} \sigma_{xx} = 0 & \tau_{xy} = -15ksi & \tau_{xz} = 0 \\ \tau_{yx} = -15ksi & \sigma_{yy} = 10ksi(C) & \tau_{yz} = 25ksi \\ \tau_{zx} = 0 & \tau_{zy} = 25ksi & \sigma_{zz} = 20ksi(T) \end{bmatrix}$$



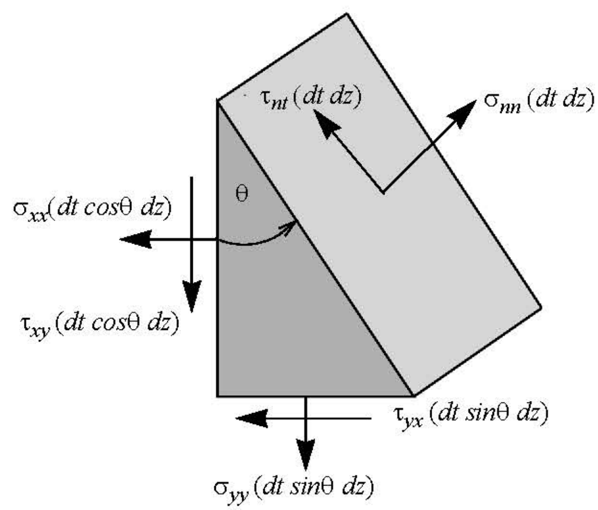
Stress transformation in two dimension



Stress Wedge



Force Wedge



$$\sigma_{nn} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{nt} = -\sigma_{xx}\cos\theta\sin\theta + \sigma_{yy}\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\sigma_{tt} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\cos\theta\sin\theta$$

Matrix Notation

$$n_x = \cos\theta \qquad n_y = \sin\theta \qquad t_x = \cos\lambda \qquad t_y = \sin\lambda$$

True only in 2D: $\lambda = 90 + \theta$; $t_x = -n_y \qquad t_y = n_x$

$$\{n\} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} \qquad \{t\} = \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} \qquad [\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$

The symmetry of shear stresses $\quad [\sigma]^T = [\sigma]$

$$\boxed{\sigma_{nn} = \{n\}^T[\sigma]\{n\}}$$

$$\boxed{\tau_{nt} = \{t\}^T[\sigma]\{n\}}$$

$$\boxed{\sigma_{tt} = \{t\}^T[\sigma]\{t\}}$$

Traction or Stress vector

Mathematically the stress vector {S} is defined as:

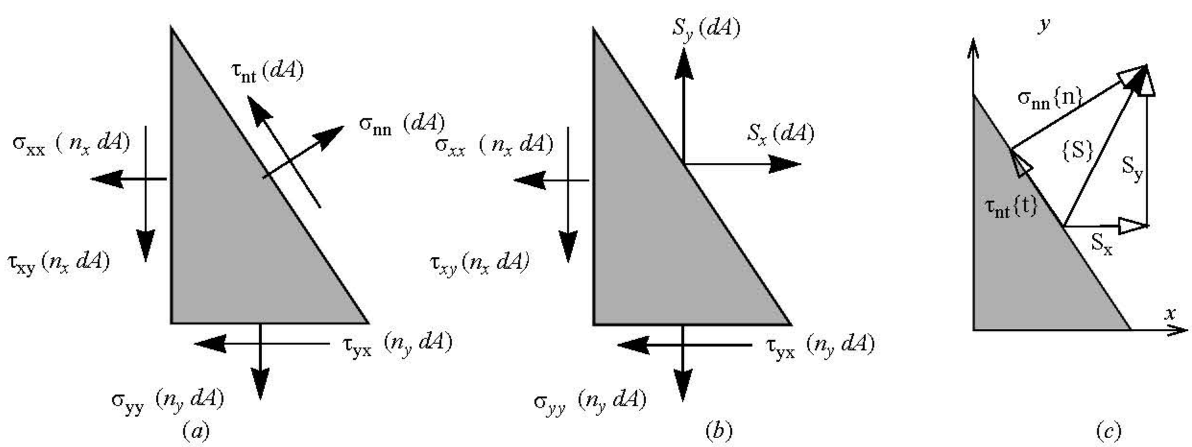
$$\{S\} = [\sigma]\{n\}$$

$$S_x = \sigma_{xx}n_x + \tau_{xy}n_y$$

$$S_y = \tau_{yx}n_x + \sigma_{yy}n_y$$

- pressure is a scaler quantity.
- traction is a vector quantity.
- stress is a second order tensor.

Statically equivalent force wedge.



Stress vector in different coordinate systems. $\{S\} = \sigma_{nn}\{n\} + \tau_{nt}\{t\}$

Principal Stresses and Directions

$$\{S\} = [\sigma]\{p\} = \sigma_p\{p\}$$

OR

$$\{S\} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = \begin{bmatrix} \sigma_p & 0 \\ 0 & \sigma_p \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

OR

$$\begin{bmatrix} (\sigma_{xx} - \sigma_p) & \tau_{xy} \\ \tau_{yx} & (\sigma_{yy} - \sigma_p) \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = 0$$

Characteristic equation

$$\sigma_p^2 - \sigma_p(\sigma_{xx} + \sigma_{yy}) + (\sigma_{xx}\sigma_{yy} - \tau_{xy}^2) = 0$$

Roots: $\sigma_{1,2} = [(\sigma_{xx} + \sigma_{yy}) \pm \sqrt{(\sigma_{xx} + \sigma_{yy})^2 - 4(\sigma_{xx}\sigma_{yy} - \tau_{xy}^2)}]/2$

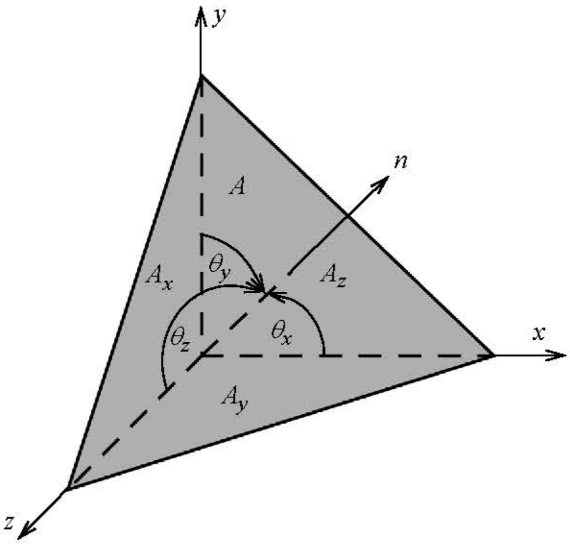
OR

$$\sigma_{1,2} = \left[\left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right) \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}\right]$$

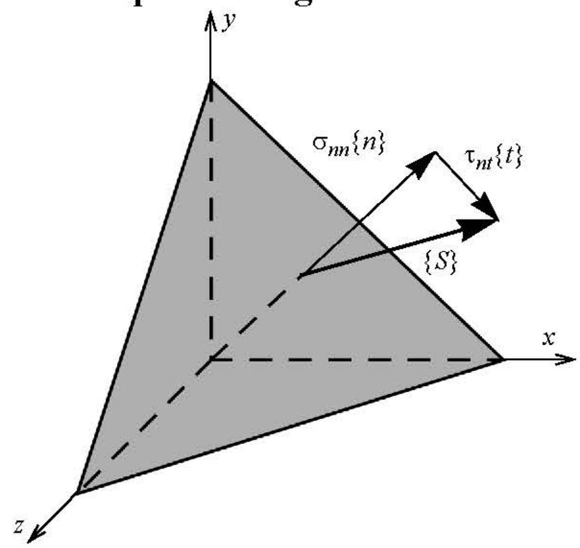
- The *eigenvalues* of the stress matrix are the principal stresses.
- The *eigenvectors* of the stress matrix are the principal directions.

Stress Transformation in 3-D

Direction cosines of a unit normal



Equilibrating shear stress



$$\{n\} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \qquad \{S\} = \begin{Bmatrix} S_x \\ S_y \\ S_z \end{Bmatrix} \qquad [\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\boxed{\sigma_{nn} = \{n\}^T[\sigma]\{n\}}$$

$$\boxed{\tau_{nt} = \{t\}^T[\sigma]\{n\}}$$

$$\boxed{\sigma_{tt} = \{t\}^T[\sigma]\{t\}}$$

$$\{S\} = [\sigma]\{n\}$$

Equilibrium condition: $\{S\} = \sigma_{nn}\{n\} + \tau_{nt}\{t_E\}$ implies $|S|^2 = \sigma_{nn}^2 + \tau_{nt}^2$

Principal Stresses and Directions

- Planes on which the shear stresses are zero are called the principal planes.
- The normal direction to the principal planes is referred to as the principal direction or the principal axis.
- The angles the principal axis makes with the global coordinate system are called the principal angles.

$$\{S\} = [\sigma]\{p\} = \sigma_p\{p\}$$

OR

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} = \begin{bmatrix} \sigma_p & 0 & 0 \\ 0 & \sigma_p & 0 \\ 0 & 0 & \sigma_p \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

OR

$$\begin{bmatrix} (\sigma_{xx} - \sigma_p) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_{yy} - \sigma_p) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_{zz} - \sigma_p) \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} = 0$$

- The *eigenvalues* of the stress matrix are the principal stresses.
- The *eigenvectors* of the stress matrix are the principal directions.

$$\boxed{p_x^2 + p_y^2 + p_z^2 = 1}$$

Principal stress convention

Ordered principal stresses in 3-D: $\sigma_1 > \sigma_2 > \sigma_3$

Ordered principal stresses in 2-D: $\sigma_1 > \sigma_2$

Principal Angles $0^o \le \theta_x, \theta_y, \theta_z \le 180^o$

Characteristic equation

$$\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$$

Stress Invariants

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \tau_{yz} \\ \tau_{zy} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \tau_{xz} \\ \tau_{zx} & \sigma_{zz} \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{vmatrix}$$

$$x^3 - I_1 x^2 + I_2 x - I_3 = 0$$

Roots:

$$x_1 = 2A\cos\alpha + I_1/3 \qquad x_{2,3} = -2A\cos(\alpha \pm 60^o) + I_1/3$$

$$A = \sqrt{(I_1/3)^2 - I_2/3}$$

$$\cos 3\alpha = [2(I_1/3)^3 - (I_1/3)I_2 + I_3]/(2A^3)$$

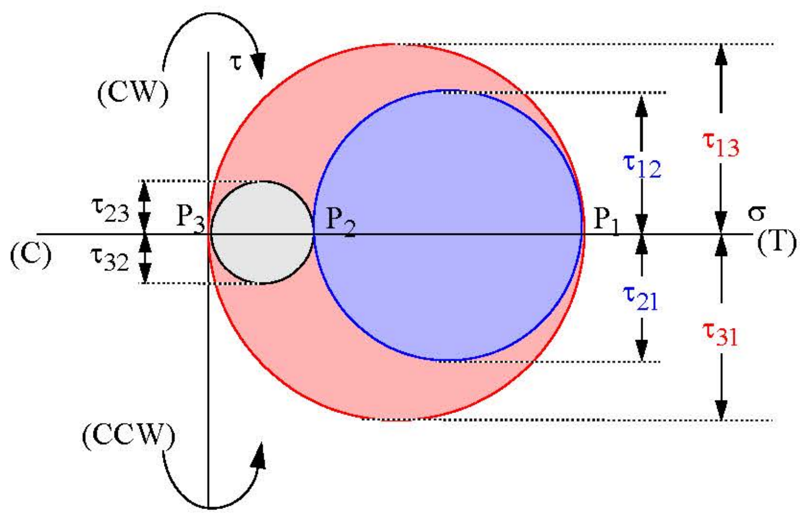
Principal Stress Matrix $[\sigma] = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$
$$I_3 = \sigma_1\sigma_2\sigma_3$$

Maximum Shear Stress

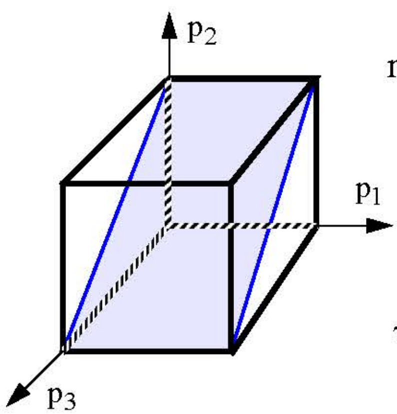
Plane Stress

$\sigma_3 = 0$



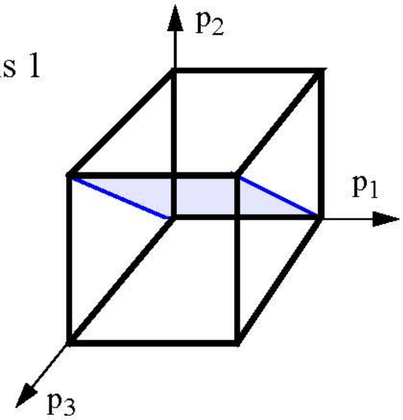
- maximum shear stress exists on two planes, each of which are 45^o away from the principal planes.

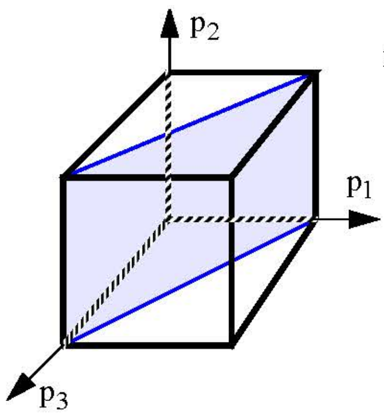
$$\tau_{max} = max\left(\left|\frac{\sigma_1 - \sigma_2}{2}\right|, \left|\frac{\sigma_2 - \sigma_3}{2}\right|, \left|\frac{\sigma_3 - \sigma_1}{2}\right|\right)$$



rotation about principal axis 1

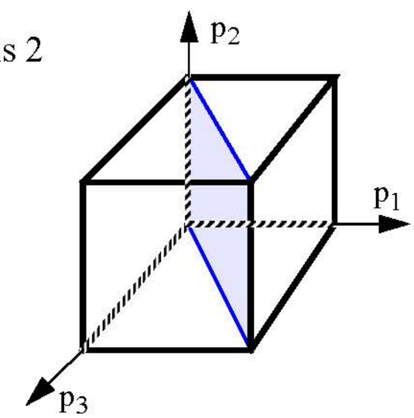
$$\tau_{23} = -\tau_{32} = \frac{\sigma_2 - \sigma_3}{2}$$

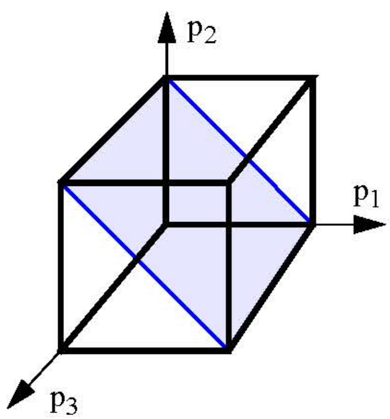




rotation about principal axis 2

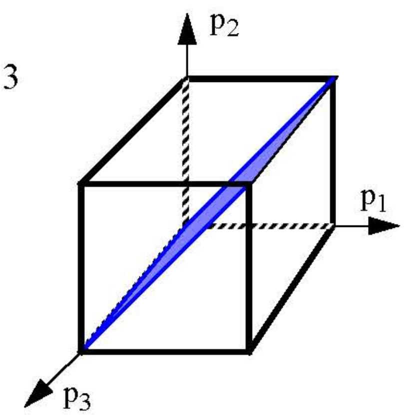
$$\tau_{31} = -\tau_{13} = \frac{\sigma_3 - \sigma_1}{2}$$





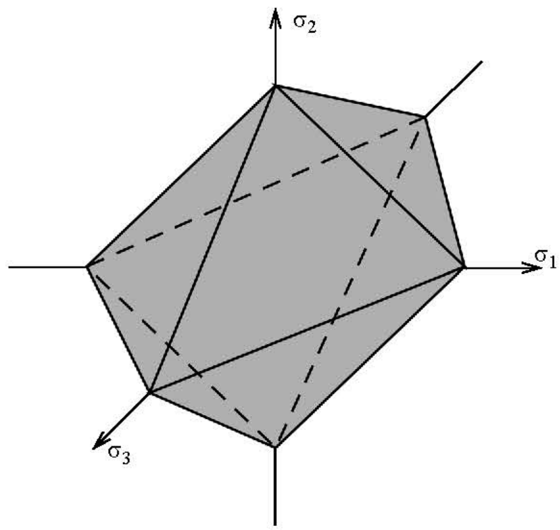
rotation about principal axis 3
(In-plane)

$$\tau_{21} = -\tau_{12} = \frac{\sigma_1 - \sigma_2}{2}$$



Octahedral stresses

- A plane that makes equal angles with the principal planes is called an octahedral plane.
- The stresses on the octahedral planes are the octahedral stresses.



$$\{S\} = \begin{Bmatrix} \sigma_1 n_1 \\ \sigma_2 n_2 \\ \sigma_3 n_3 \end{Bmatrix}$$

$$\sigma_{nn} = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$$

$$\tau_{nt} = \sqrt{|S|^2 - \sigma_{nn}^2} = \sqrt{(\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2) - \sigma_{nn}^2}$$

$$|n_1| = |n_2| = |n_3| = 1/\sqrt{3}$$

$$\sigma_{oct} = (\sigma_1 + \sigma_2 + \sigma_3)/3 = I_1/3$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

Stress Deviators

Experiments have shown that hydrostatic pressure has negligible effect on the yield point until extreme high pressures are reached[1] (> 360 ksi). The high hydrostatic pressure does not effect the stress-strain curve in the elastic region but increase the ductility of the material, i.e., permits large plastic deformation before fracture.

Stress deviatoric matrix is the stress matrix from which the hydrostatic state of stress has been removed. The hydrostatic pressure (p) is given by

$$p = \frac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} = \frac{\sigma_1+\sigma_2+\sigma_3}{3} = \frac{I_1}{3}$$

where, I_1 is the first stress invariant.

The stress deviatoric matrix in Cartesian coordinate principal coordinates is given by

Stress deviatoric matrices

$$\begin{bmatrix} \sigma_{xx}-\frac{I_1}{3} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}-\frac{I_1}{3} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}-\frac{I_1}{3} \end{bmatrix} \qquad \begin{bmatrix} \sigma_1 - I_1/3 & 0 & 0 \\ 0 & \sigma_2 - I_1/3 & 0 \\ 0 & 0 & \sigma_3 - I_1/3 \end{bmatrix}$$

The deviatoric stress invariants are as given below

$$J_1 = 0$$

$$J_2 = \begin{vmatrix} \sigma_1 - I_1/3 & 0 \\ 0 & \sigma_2 - I_1/3 \end{vmatrix} + \begin{vmatrix} \sigma_2 - I_1/3 & 0 \\ 0 & \sigma_3 - I_1/3 \end{vmatrix} + \begin{vmatrix} \sigma_1 - I_1/3 & 0 \\ 0 & \sigma_3 - I_1/3 \end{vmatrix}$$

$$J_2 = I_2 - \frac{1}{3}I_1^2 = -\left(\frac{1}{6}\right)[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2] = -\left(\frac{3}{2}\right)\tau_{oct}^2$$

$$J_3 = I_3 - \frac{1}{3}I_1I_2 + \frac{2}{27}I_1^3 = \frac{1}{27}(2\sigma_1-\sigma_2-\sigma_3)(2\sigma_2-\sigma_3-\sigma_1)(2\sigma_3-\sigma_1-\sigma_2)$$

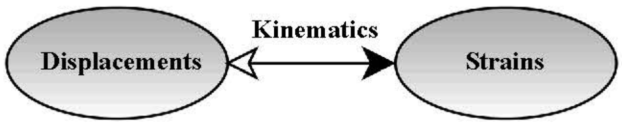
1. Mendelson A., "Plasticity: Theory and Applications", Macmillan Co., New York, (1968) section 2-5.

C1.2 The stress at a point is given by the stress matrix shown. Determine: (a) the normal and shear stress on a plane that has an outward normal at 37^o, 120^o, and 70.43^o, to x, y, and z direction respectively. (b) the principal stresses (c) the second principal direction and (d) the magnitude of the octahedral shear stress. (e) maximum shear stress (f) the deviatoric stress invariants.

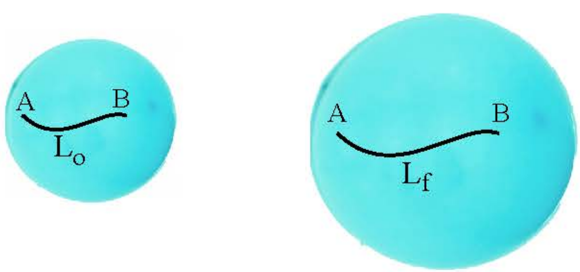
$$\begin{bmatrix} 18 & 12 & 9 \\ 12 & 12 & -6 \\ 9 & -6 & 6 \end{bmatrix} ksi$$

Strain

- The total movement of a point with respect to a fixed reference coordinates is called *displacement*.
- The relative movement of a point with respect to another point on the body is called *deformation*.
- *Lagrangian strain* is computed from deformation by using the original undeformed geometry as the reference geometry.
- *Eulerian strain* is computed from deformation by using the final deformed geometry as the reference geometry.
- Relating strains to displacements is a problem in geometry.



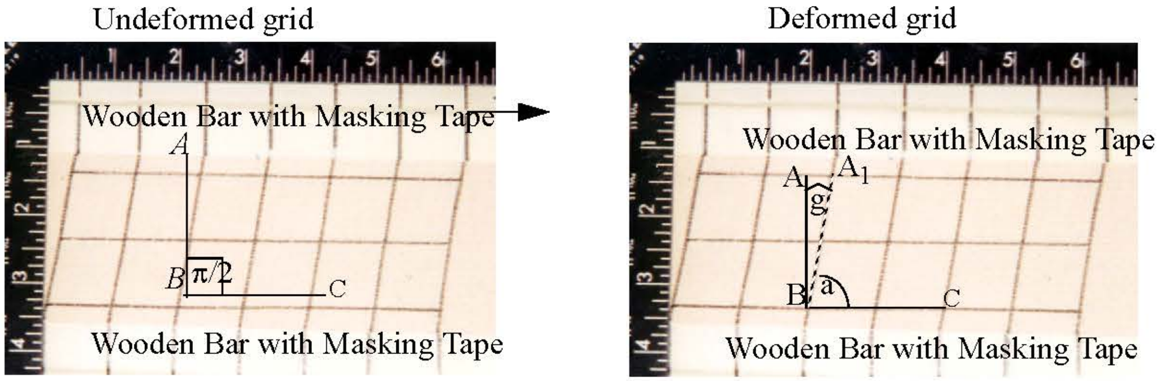
Average normal strain



$$\varepsilon_{av} = \frac{L_f - L_o}{L_o} = \frac{\delta}{L_o}$$

- Elongations ($L_f > L_o$) result in *positive* normal strains. Contractions ($L_f < L_o$) result in *negative* normal strains.

Average shear strain



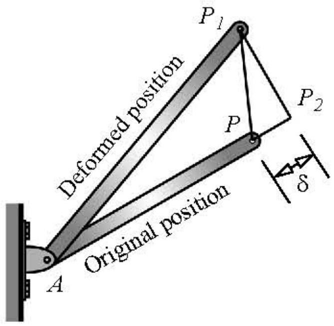
$$\gamma_{av} = \frac{\pi}{2} - \alpha$$

- Decreases in the angle ($\alpha < \pi / 2$) result in positive shear strain. Increase in the angle ($\alpha > \pi / 2$) result in negative shear strain

Units of average strain

- To differentiate average strain from strain at a point.
- in/in, or cm/cm, or m/m (for normal strains)
- rads (for shear strains)
- percentage. 0.5% is equal to a strain of 0.005
- prefix: $\mu = 10^{-6}$. 1000 μ in / in is equal to a strain 0.001 in /

Small Strain Approximation



$$L_f = \sqrt{L_o^2 + D^2 + 2L_o D\cos\theta}$$

$$L_f = L_o\sqrt{1 + \left(\frac{D}{L_o}\right)^2 + 2\left(\frac{D}{L_o}\right)\cos\theta}$$

$$\varepsilon = \frac{L_f - L_o}{L_o} = \sqrt{1 + \left(\frac{D}{L_o}\right)^2 + 2\left(\frac{D}{L_o}\right)\cos\theta} - 1$$

$$\varepsilon_{small} = \frac{D\cos\theta}{L_o}$$

ε_{small}	ε	% error
1.0	1.23607	19.1
0.5	0.58114	14.0
0.1	0.10454	4.3
0.05	0.005119	2.32
0.01	0.01005	0.49
0.005	0.00501	0.25

- Small-strain approximation may be used for strains less than 0.01
- Small normal strains are calculated by using the deformation component in the original direction of the line element regardless of the orientation of the deformed line element.
- In small shear strain (γ) calculations the following approximation may be used for the trigonometric functions: $\tan\gamma \approx \gamma$ $\sin\gamma \approx \gamma$ $\cos\gamma \approx 1$
- Small-strain calculations result in linear deformation analysis.
- Drawing approximate deformed shape is very important in analysis of small strains.

C1.3 A roller at P slides in a slot as shown. Determine the deformation in bar AP and bar BP by using small strain approximation.

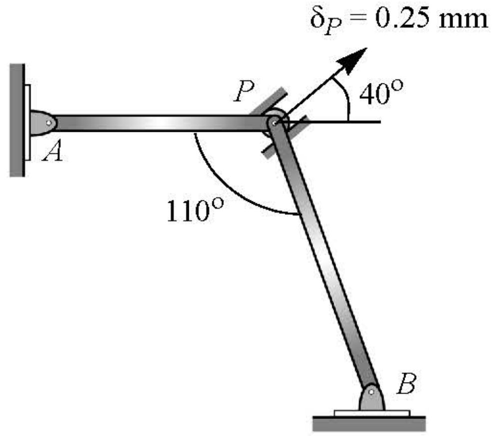
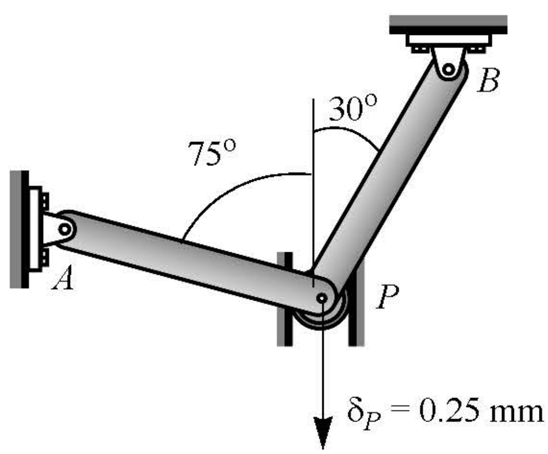


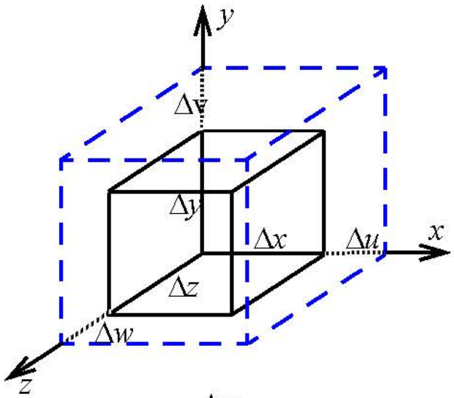
Fig. C1.3

Class Problem 1.2

Draw an approximate exaggerated deformed shape.
Using small strain approximation write equations relating δ_{AP} and δ_{BP} to δ_P.



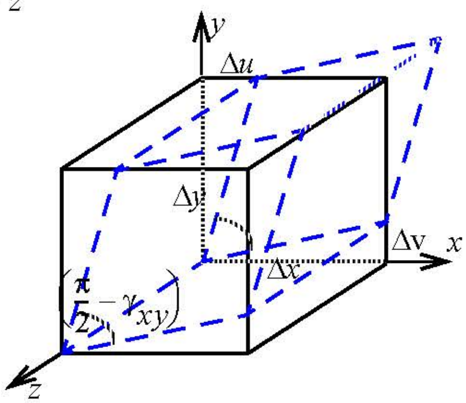
Engineering strain at a point



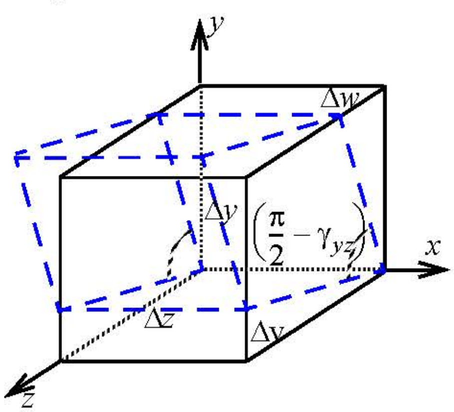
$$\varepsilon_{xx} = \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x}\right) = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \lim_{\Delta y \to 0}\left(\frac{\Delta v}{\Delta y}\right) = \frac{\partial v}{\partial y}$$

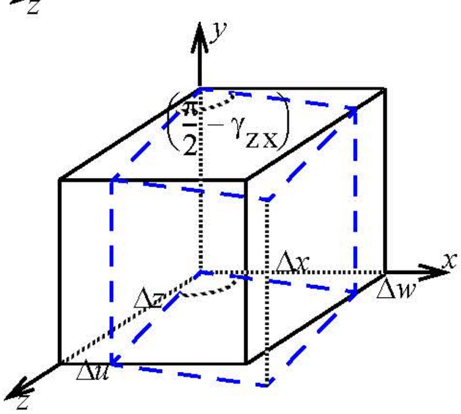
$$\varepsilon_{zz} = \lim_{\Delta z \to 0}\left(\frac{\Delta w}{\Delta z}\right) = \frac{\partial w}{\partial z}$$



$$\gamma_{xy} = \gamma_{yx} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}}\left(\frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x}\right) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



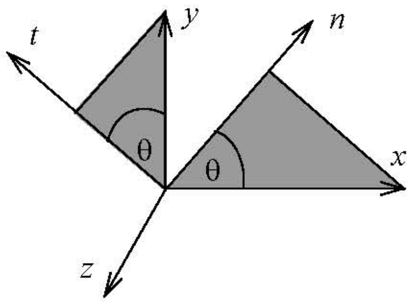
$$\gamma_{yz} = \gamma_{zy} = \lim_{\substack{\Delta y \to 0 \\ \Delta z \to 0}}\left(\frac{\Delta v}{\Delta z} + \frac{\Delta w}{\Delta y}\right) = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$



$$\gamma_{zx} = \gamma_{xz} = \lim_{\substack{\Delta x \to 0 \\ \Delta z \to 0}}\left(\frac{\Delta w}{\Delta x} + \frac{\Delta u}{\Delta z}\right) = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Strain Transformation

Strain transformation is relating strains in two coordinate systems.



$$x = n\cos\theta - t\sin\theta \qquad y = n\sin\theta + t\cos\theta$$

$$u_n = u\cos\theta + v\sin\theta \qquad v_t = -u\sin\theta + v\cos\theta$$

$$\varepsilon_{nn} = \frac{\partial u_n}{\partial n} = \frac{\partial u_n}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial u_n}{\partial y}\frac{\partial y}{\partial n}$$

Strain transformation equations in 2-D

$$\varepsilon_{nn} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$

$$\varepsilon_{tt} = \varepsilon_{xx}\sin^2\theta + \varepsilon_{yy}\cos^2\theta - \gamma_{xy}\sin\theta\cos\theta$$

$$\gamma_{nt} = -2\varepsilon_{xx}\sin\theta\cos\theta + 2\varepsilon_{yy}\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta)$$

Stress transformation equations in 2-D

$$\sigma_{nn} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_{tt} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\cos\theta\sin\theta$$

$$\tau_{nt} = -\sigma_{xx}\cos\theta\sin\theta + \sigma_{yy}\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

- tensor normal strains = engineering normal strains
- tensor shear strains = (engineering shear strains)/ 2

Tensor strain matrix from engineering strains

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} = \gamma_{xy}/2 & \varepsilon_{xz} = \gamma_{xz}/2 \\ \varepsilon_{yx} = \gamma_{yx}/2 & \varepsilon_{yy} & \varepsilon_{yz} = \gamma_{yz}/2 \\ \varepsilon_{zx} = \gamma_{zx}/2 & \varepsilon_{zy} = \gamma_{zy}/2 & \varepsilon_{zz} \end{bmatrix}$$

$$\varepsilon_{nn} = \{n\}^T[\varepsilon]\{n\}$$

$$\varepsilon_{nt} = \{t\}^T[\varepsilon]\{n\} \qquad \gamma_{nt} = 2\varepsilon_{nt}$$

$$\varepsilon_{tt} = \{t\}^T[\varepsilon]\{t\}$$

Characteristic equation

$$\varepsilon_p^3 - I_1\varepsilon_p^2 + I_2\varepsilon_p - I_3 = 0$$

Strain invariants

$$I_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

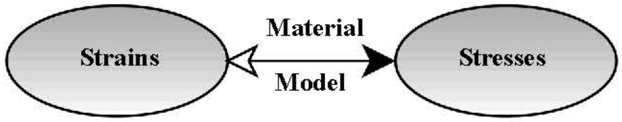
$$I_2 = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{vmatrix} + \begin{vmatrix} \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zy} & \varepsilon_{zz} \end{vmatrix} + \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xz} \\ \varepsilon_{zx} & \varepsilon_{zz} \end{vmatrix} = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1$$

$$I_3 = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{vmatrix} = \varepsilon_1\varepsilon_2\varepsilon_3$$

Maximum shear strain

$$\frac{\gamma_{max}}{2} = max\left(\left|\frac{\varepsilon_1 - \varepsilon_2}{2}\right|, \left|\frac{\varepsilon_2 - \varepsilon_3}{2}\right|, \left|\frac{\varepsilon_3 - \varepsilon_1}{2}\right|\right)$$

Material Description



Linear Material Models

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

$$C_{ij} = C_{ji}$$

- The most general linear anisotropic material requires 21 independent constants.

Monoclinic material

- Has 1 plane of symmetry.
- If xy is the plane of symmetry then stress-strain relations in +ve & -ve z direction are the same.
- Requires 13 independent material constants.

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

- x, y, z are the *material coordinate system.*
- The zero's in the C matrix can become non-zero in coordinate systems other than *material coordinate system.*

Orthotropic material

- Has two axis of symmetry.
- Requires 9 independent constants in 3_D.

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

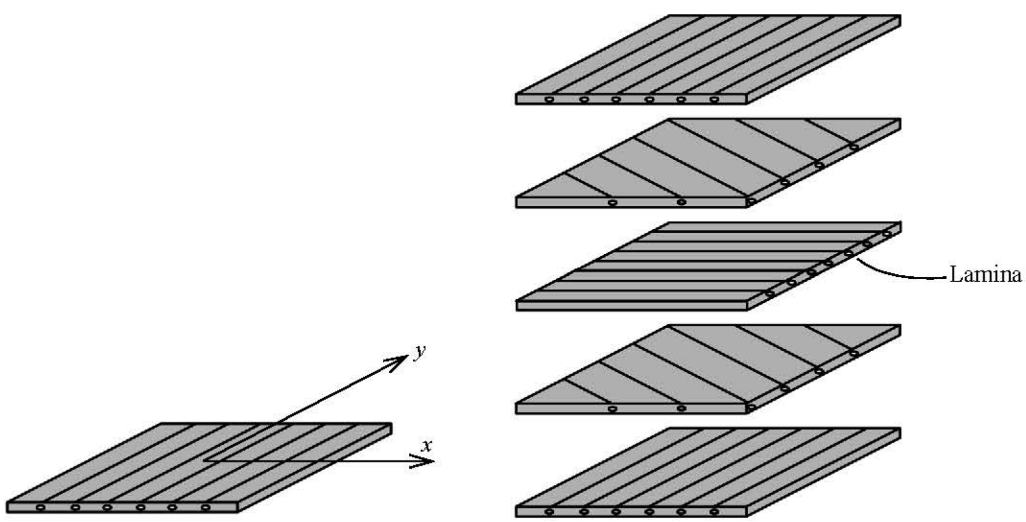
- *x, y, z* are the *material coordinate system.*
- The zero's in the C matrix can become non-zero in coordinate systems other than *material coordinate system.*

For plane stress problems (requires 4 independent constants)

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_x} - \frac{\nu_{yx}}{E_y}\sigma_{yy} \qquad \varepsilon_{yy} = \frac{\sigma_{yy}}{E_y} - \frac{\nu_{xy}}{E_x}\sigma_{xx} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \qquad \frac{\nu_{yx}}{E_y} = \frac{\nu_{xy}}{E_x}$$

Long Fiber Composite

- Each lamina is an orthotropic material.
- A symmetric stacking about mid surface creates an orthotropic composite plate.



Transversely isotropic material

- Material is isotropic in a plane.
- Requires 5 independent material constants.

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} C_{12} C_{13} & 0 & 0 & 0 \\ C_{12} C_{11} C_{13} & 0 & 0 & 0 \\ C_{13} C_{13} C_{33} & 0 & 0 & 0 \\ 0\ \ 0\ \ 0 & C_{44} & 0 & 0 \\ 0\ \ 0\ \ 0 & 0 & C_{44} & \\ 0\ \ 0\ \ 0 & 0 & 0 & 2(C_{11}-C_{12}) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

- The zero's in the C matrix can become non-zero in coordinate systems other than *material coordinate system*.

Short Fiber Composite

Chopped fiber is sprayed on to a epoxy produces a transversely isotropic material. It is isotropic in the plane.

Isotropic Material

- An isotropic material has a stress-strain relationships that are independent of the orientation of the coordinate system at a point.
- An isotropic body requires only two independent material constants

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} C_{12} C_{12} & 0 & 0 & 0 \\ C_{12} C_{11} C_{12} & 0 & 0 & 0 \\ C_{12} C_{12} C_{11} & 0 & 0 & 0 \\ 0\ \ 0\ \ 0 & 2(C_{11}-C_{12}) & 0 & 0 \\ 0\ \ 0\ \ 0 & 0 & 2(C_{11}-C_{12}) & \\ 0\ \ 0\ \ 0 & 0 & 0 & 2(C_{11}-C_{12}) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

Engineering Constants: $C_{11} = 1/E$, $C_{12} = -\nu/E$, and $2(C_{11}-C_{12}) = 1/G$

- E = Modulus of Elasticity
- G = Shear Modulus of Elasticity

- ν = Poisson's Ratio

Generalized Hooke's Law

$$\varepsilon_{xx} = [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]/E \qquad \gamma_{xy} = \tau_{xy}/G$$
$$\varepsilon_{yy} = [\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})]/E \qquad \gamma_{yz} = \tau_{yz}/G \qquad G = \frac{E}{2(1+\nu)}$$
$$\varepsilon_{zz} = [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]/E \qquad \gamma_{zx} = \tau_{zx}/G$$

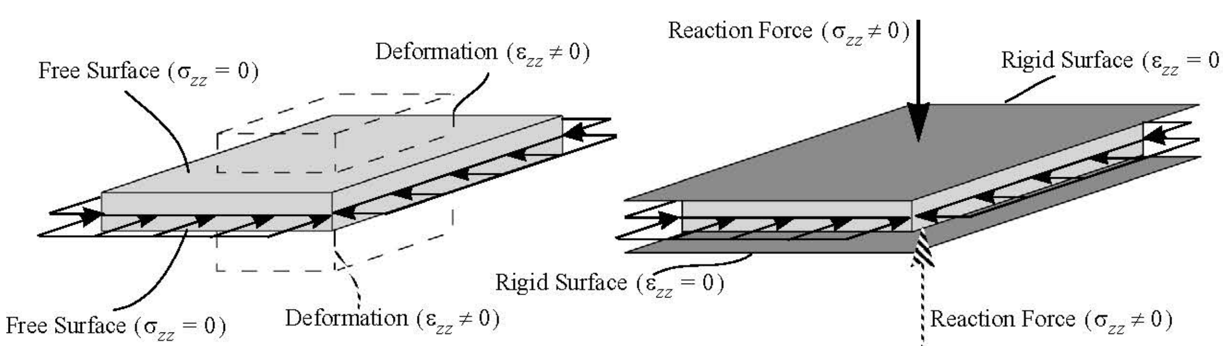
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}$$

- Generalized Hooke's Law is valid for any orthogonal coordinate system.
- Principal direction for stress and strain are same *ONLY* for isotropic materials.
- A material is said to be homogeneous if the material properties are the same at all points on the body. Alternatively, if the material constants C_{ij} are functions of the coordinates x, y, or z, then the material is called non-homogeneous.

Plane Stress and Plane Strain.

$$\text{Plane Stress} \longrightarrow \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Generalized Hooke's Law}} \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \end{bmatrix}$$

$$\text{Plane Strain} \longrightarrow \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Generalized Hooke's Law}} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \end{bmatrix}$$

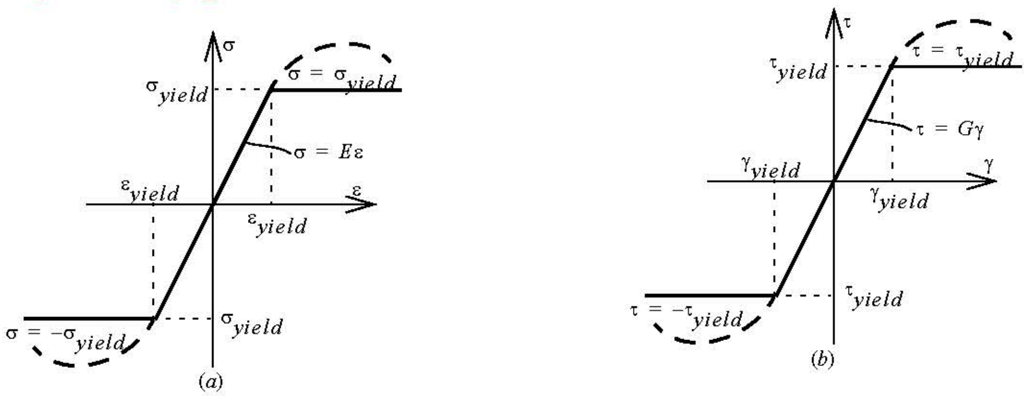


Non-linear material models

- Elastic-perfectly plastic in which the non-linearity is approximated by a constant.
- Linear strain hardening model (Bi-linear model) in which the non-linearity is approximated by a linear function.
- Power law model in which the non-linearity is approximated by one term non-linear function.

We will assume material behavior is same in tension and compression.

Elastic-perfectly plastic

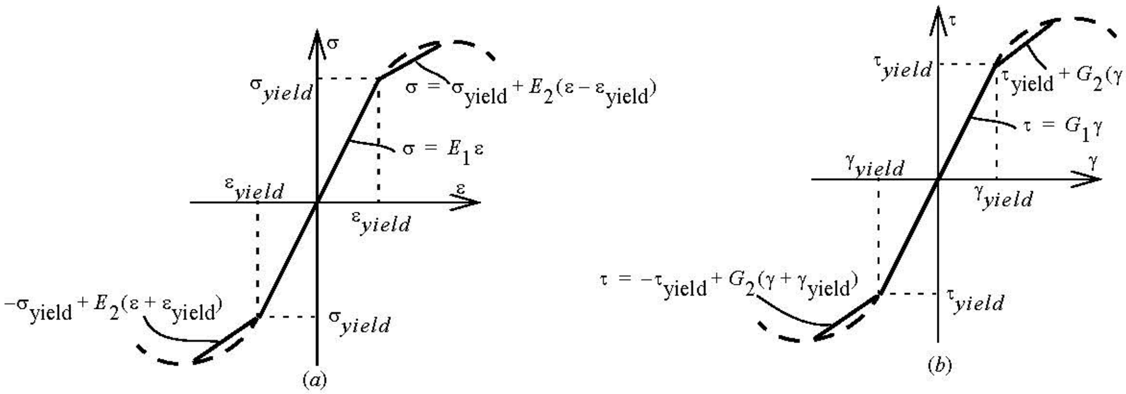


$$\sigma = \begin{cases} \sigma_{yield} & \varepsilon \geq \varepsilon_{yield} \\ E\varepsilon & -\varepsilon_{yield} \leq \varepsilon \leq \varepsilon_{yield} \\ -\sigma_{yield} & \varepsilon \leq -\varepsilon_{yield} \end{cases} \qquad \tau = \begin{cases} \tau_{yield} & \gamma \geq \gamma_{yield} \\ G\gamma & -\gamma_{yield} \leq \gamma \leq \gamma_{yield} \\ -\tau_{yield} & \gamma \leq -\gamma_{yield} \end{cases}$$

- The set of points forming the boundary between the elastic and plastic region on a body, is called the elastic-plastic boundary.

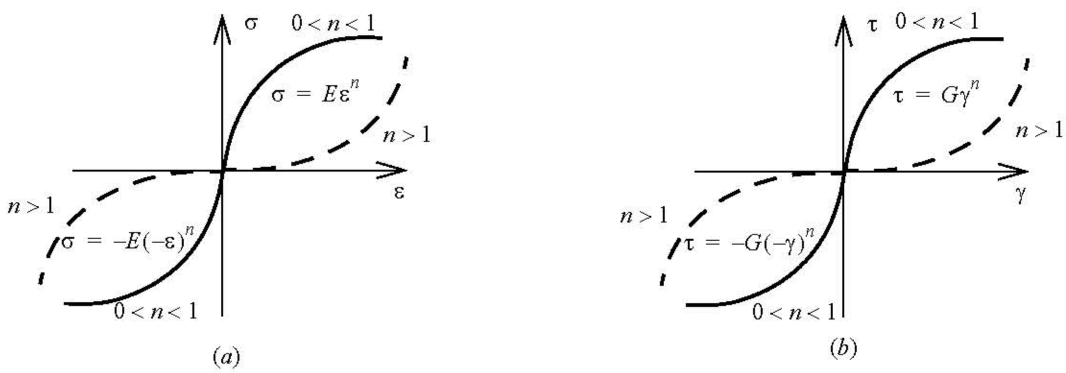
1. On the elastic-plastic boundary the strain must be equal to the yield strain, and stress equal to yield stress.
2. Deformations and strains are continuous at all points including points at the elastic plastic boundary.
3. In beam bending, the location of neutral axis depends material property, geometry, and loading.

Linear strain hardening material model



$$\sigma = \begin{cases} \sigma_{yield} + E_2(\varepsilon - \varepsilon_{yield}) & \varepsilon \geq \varepsilon_{yield} \\ E_1\varepsilon & -\varepsilon_{yield} \leq \varepsilon \leq \varepsilon_{yield} \\ -\sigma_{yield} + E_2(\varepsilon + \varepsilon_{yield}) & \varepsilon \geq \varepsilon_{yield} \end{cases}$$

Power Law

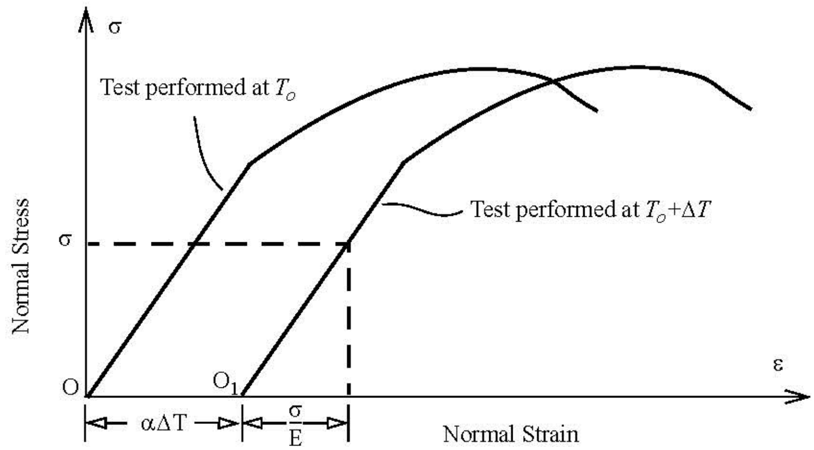


$$\sigma = \begin{cases} E\varepsilon^n & \varepsilon \geq 0 \\ -E(-\varepsilon)^n & \varepsilon < 0 \end{cases}$$

$n < 1$ —metals and plastics

$n > 1$ —soft rubber, muscles, and organic materials.

Effects of Temperature



$$\varepsilon = \frac{\sigma}{E} + \alpha\Delta T$$

α is linear coefficient of thermal expansion that has units of $\mu/^{o}F$ or $\mu/^{o}C$

- No thermal stresses are produced in a homogeneous, isotropic, unconstrained body due to uniform temperature changes.

$$\varepsilon_{xx} = [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]/E + \alpha\ \Delta T$$
$$\varepsilon_{yy} = [\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})]/E + \alpha\ \Delta T$$
$$\varepsilon_{zz} = [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]/E + \alpha\ \Delta T$$
$$\gamma_{xy} = \tau_{xy}/G$$
$$\gamma_{yz} = \tau_{yz}/G$$
$$\gamma_{zx} = \tau_{zx}/G$$

Thermal Strain

Mechanical Strain

C1.4 The stress at a point, material properties, and change in temperature are as given below. Calculate ε_{xx}, ε_{yy}, γ_{xy}, ε_{zz}, and σ_{zz} (a) assuming plane stress, and (b) assuming plane strain.

$$\sigma_{xx} = 300\ MPa(C) \qquad \sigma_{yy} = 300\ MPa(T) \qquad \tau_{xy} = 150\ MPa$$

$$G = 15\ GPa \qquad \nu = 0.2 \qquad \alpha = 26.0\mu/^{o}C \qquad \Delta T = 75^{o}C$$

Failure Theories

- A failure theory is a statement on relationship of the stress components to material failure characteristics values.

	Ductile Material	Brittle Material
Characteristic failure stress	Yield stress	Ultimate stress
Theories	1. Maximum shear stress 2. Maximum octahedral shear stress	1. Maximum normal stress 2. Modified Mohr

Maximum shear stress theory

For ductile materials the theory predicts

A material will fail when the maximum shear stress exceeds the shear stress at yield that is obtained from uniaxial tensile test.

The failure criterion is

$$\tau_{max} \le \tau_{yield}$$

$$\boxed{max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) \le \sigma_{yield}}$$

Maximum octahedral shear stress theory (Maximum distortion strain energy or von-Mises criterion)

For ductile materials the theory predicts

A material will fail when the maximum octahedral shear stress exceeds the octahedral shear stress at yield that is obtained from uniaxial tensile test.

The failure criterion is

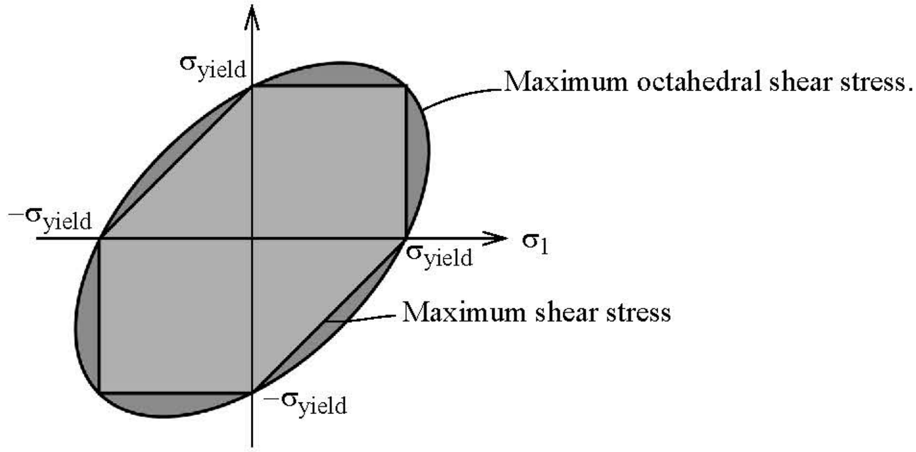
$$\tau_{oct} \le \overline{\tau_{yield}}$$

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \le \sigma_{yield}$$

Equivalent von-Mises Stress

$$\boxed{\sigma_{von} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}} \qquad \boxed{\sigma_{von} \le \sigma_{yield}}$$

Failure Envelopes for ductile materials in plane stress



Maximum normal stress theory

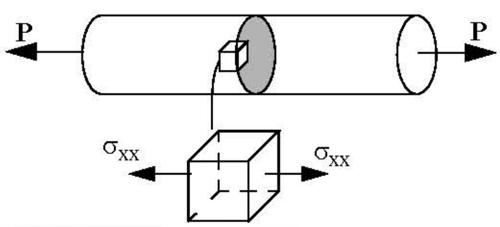
For brittle materials the theory predicts

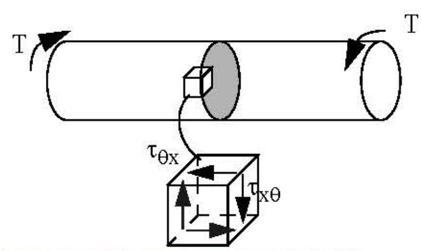
A material will fail when the maximum normal stress at a point exceed the ultimate normal stress (σ_{ult}) obtained from uniaxial tension test.

$$\boxed{max(\sigma_1, \sigma_2, \sigma_3) \le \sigma_{ult}}$$

- can be used *if principal stress one is tensile* and the *dominant principal* stress.

Examples of brittle and ductile material failure





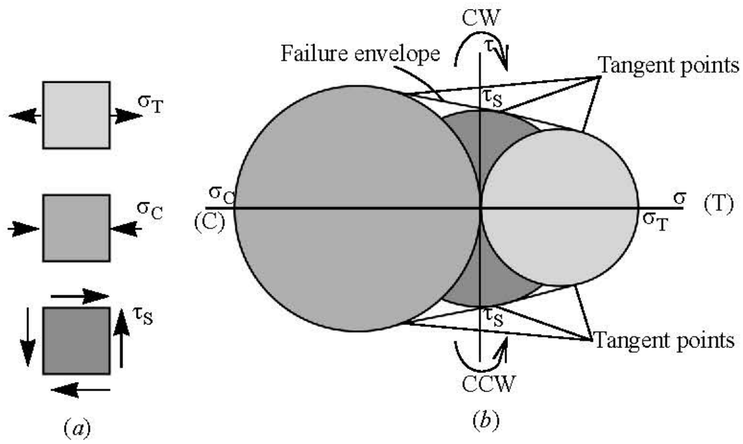
Cast Iron

Aluminum

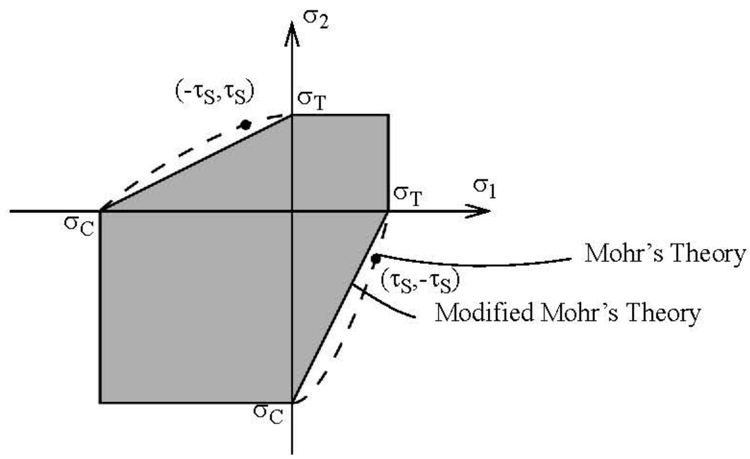
Mohr's theory

For brittle materials the theory predicts

A material will fail if a stress state is on the envelope that is tangent to the three Mohr's circles corresponding to: uniaxial ultimate stress in tension, to uniaxial ultimate stress in compression, and to pure shear.



Modified Mohr's Theory



- If both principal stresses are tensile than the maximum normal stress has to be less than the ultimate tensile strength.
- If both principal stresses are negative than the maximum normal stress must be less than the ultimate compressive strength.
- If the principal stresses are of different signs then for the Modified Mohr's Theory the failure is governed by

$$\left|\frac{\sigma_2}{\sigma_C} - \frac{\sigma_1}{\sigma_T}\right| \le 1$$

- σ_C is the magnitude of the compressive strength.

C1.5 On a free surface of aluminum (E = 10,000 ksi, ν = 0.25, σ_{yield} = 24 ksi), the strains recorded by the three strain gages shown below are ε_a = –600 μ in/in, ε_b = 500 μ in/in, and ε_c = 400 μ in/in. By how much can the loads be scaled without exceeding the yield stress of aluminum at the point? Use the maximum shear stress theory.