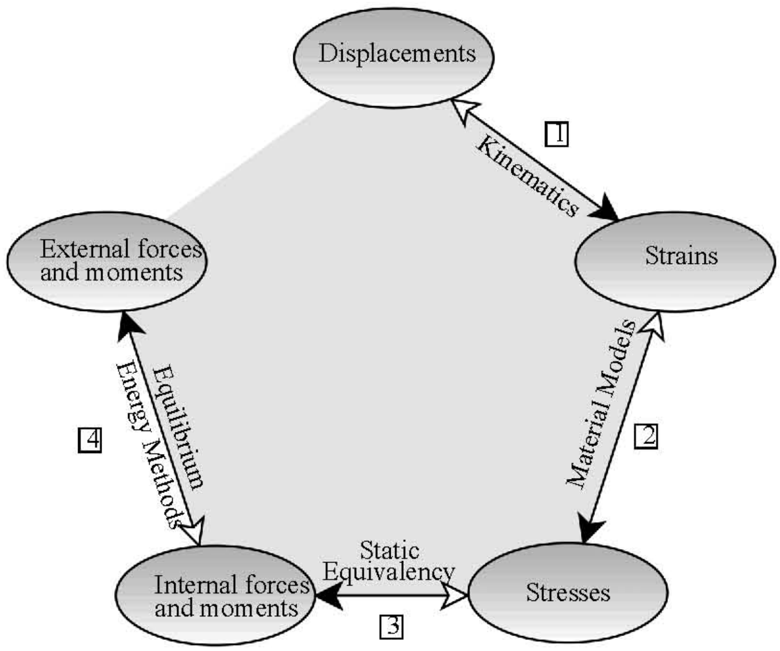
Thin-Walled Structural Members

The learning objectives of this chapter are:

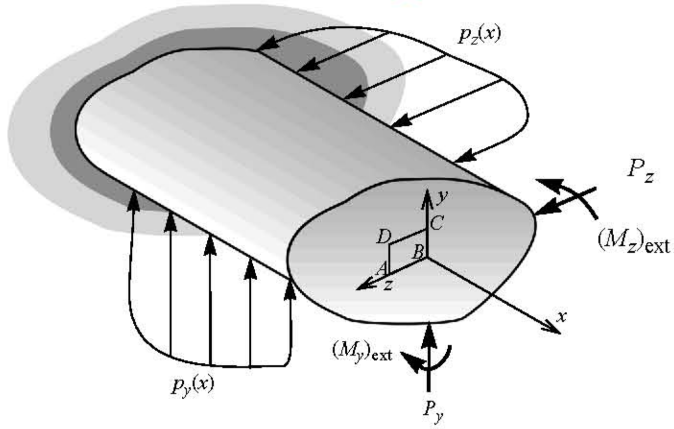
- Understand the theory, its limitations, and its application in design and analysis of unsymmetric bending of beam.
- Understand the concept of shear center and how to determine its location.

Pick up point in logic for unsymmetric bending.

- Drop the limitation that the beam has a plane of symmetry and the loading is in the plane of symmetry. This changes the kinematic relationship between displacements and strains
- Assume loading is such that there is no twisting of the cross-section.



Theory



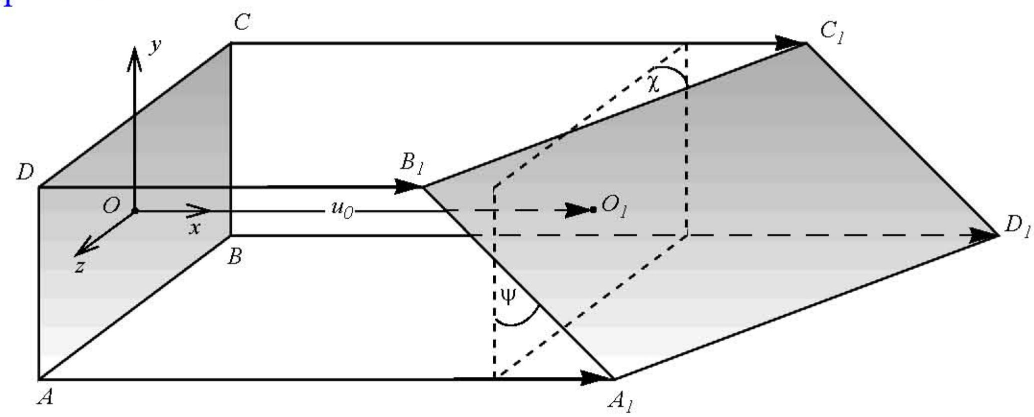
Theory objective is:

- Relate the internal shear forces V_y, V_z and internal moment M_y, M_z to displacements v and w and obtain the stresses in unsymmetric bending.

Deformation Behavior

Assumption 1 Deformations are not functions of time.

Assumption 2a The loads are such that there is no axial or torsional deformation.



No twist implies: $\gamma_{yz} = \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} = 0$

$$v(x, y, z) = v(x, y) \qquad w(x, y, z) = w(x, z)$$

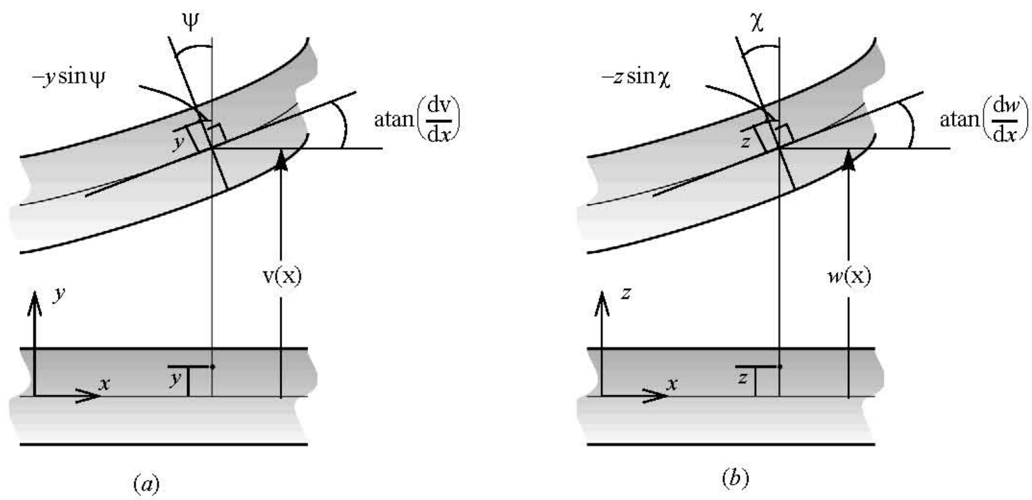
Assumption 2b Squashing action is significantly smaller then bending action.

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \approx 0 \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} \approx 0$$

$$v = v(x) \qquad w = w(x)$$

Assumption 2c Plane sections before deformation remain plane after deformation.

$$u = u_o - \psi y - \chi z$$



Assumption 2d Plane perpendicular to the axis remain nearly perpendicular after deformation.

$$u = -y\frac{\mathrm{d}v}{\mathrm{d}x} - z\frac{\mathrm{d}w}{\mathrm{d}x}$$

Strain Distribution

Assumption 3 The strains are small.

$$\boxed{\varepsilon_{xx} = \frac{\mathrm{d}u}{\mathrm{d}x} = -y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}}$$

Material Model

Assumption 4 Material is isotropic.

Assumption 5 There are no inelastic strain.

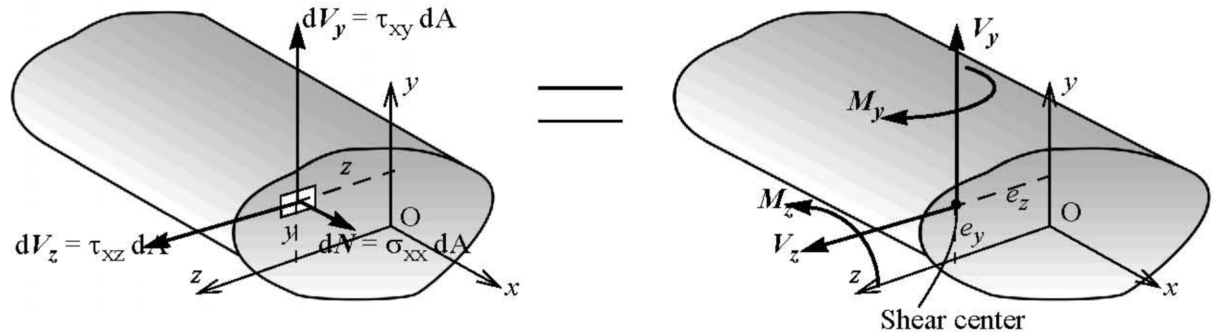
Assumption 6 Material is elastic.

Assumption 7 Stress and strains are linearly related

Hooke's Law: $\sigma_{xx} = E\varepsilon_{xx}$

$$\sigma_{xx} = -Ey\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - Ez\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$$

Internal forces and moments



$$N = \int_A \sigma_{xx} dA = 0$$

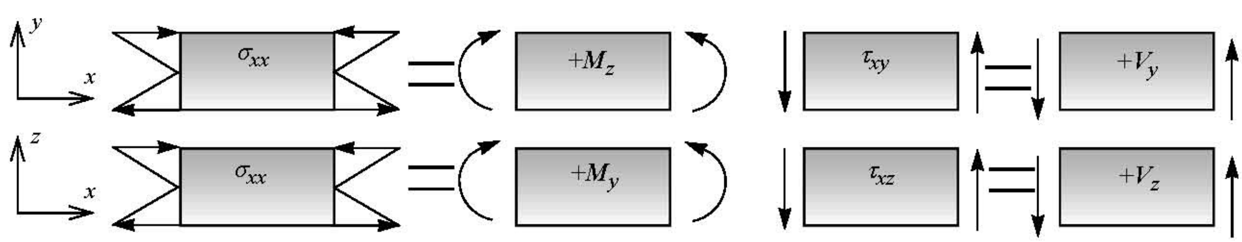
$$M_z = -\int_A y\sigma_{xx} dA \qquad M_y = -\int_A z\sigma_{xx} dA$$

$$V_y = \int_A \tau_{xy} dA \qquad V_z = \int_A \tau_{xz} dA$$

$$T = \int_A [(y - e_y)\tau_{xz} - (z - e_z)\tau_{xy}] dA = 0$$

- The maximum normal stress σ_{xx} in the beam should be nearly an order of magnitude (factor of 10) greater than the maximum shear stress τ_{xy} and τ_{xz}.

Sign Convention



Bending Formulas

Substituting $\sigma_{xx} = -Ey\dfrac{d^2 v}{dx^2} - Ez\dfrac{d^2 w}{dx^2}$ into internal moment expression.

$$\boldsymbol{M}_z = \frac{d^2 v}{dx^2}\int_A Ey^2 dA + \frac{d^2 w}{dx^2}\int_A EyzdA \qquad \boldsymbol{M}_y = \frac{d^2 v}{dx^2}\int_A EyzdA + \frac{d^2 w}{dx^2}\int_A Ez^2 dA$$

Assumption 8 Material is homogenous across the cross-section.

$$\boldsymbol{M}_z = EI_{zz}\frac{d^2 v}{dx^2} + EI_{yz}\frac{d^2 w}{dx^2} \qquad \boldsymbol{M}_y = EI_{yz}\frac{d^2 v}{dx^2} + EI_{yy}\frac{d^2 w}{dx^2}$$

Area moment of inertia

$$I_{zz} = \int_A y^2 dA \qquad I_{yy} = \int_A z^2 dA \qquad I_{yz} = \int_A yzdA$$

Moment Curvature Relationship

$$\boxed{\frac{d^2 v}{dx^2} = \frac{1}{E}\left(\frac{I_{yy}\boldsymbol{M}_z - I_{yz}\boldsymbol{M}_y}{I_{yy}I_{zz} - I_{yz}^2}\right)} \qquad \boxed{\frac{d^2 w}{dx^2} = \frac{1}{E}\left(\frac{I_{zz}\boldsymbol{M}_y - I_{yz}\boldsymbol{M}_z}{I_{yy}I_{zz} - I_{yz}^2}\right)}$$

Stress Formula

$$\boxed{\sigma_{xx} = -\left(\frac{I_{yy}\boldsymbol{M}_z - I_{yz}\boldsymbol{M}_y}{I_{yy}I_{zz} - I_{yz}^2}\right)y - \left(\frac{I_{zz}\boldsymbol{M}_y - I_{yz}\boldsymbol{M}_z}{I_{yy}I_{zz} - I_{yz}^2}\right)z}$$

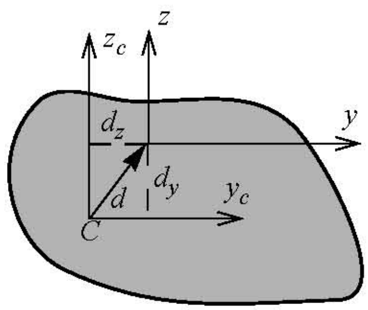
Location of origin

Centroid: $\int_A ydA = 0 \qquad \int_A zdA = 0$

- For a linear-elastic-homogeneous material cross section in unsymmetric bending the origin is at the centroid of the cross section.
- Normal stress σ_{xx} in bending varies linearly with y and z on a homogenous cross-section.

Second Area Moment of Inertias

Parallel axis theorem



$$I_{yy} = I_{y_c y_c} + A d_z^2$$

$$I_{zz} = I_{z_c z_c} + A d_y^2$$

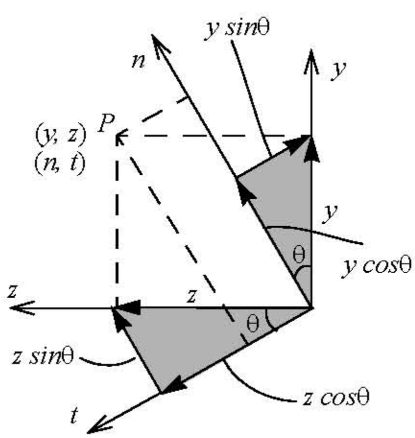
$$I_{yz} = I_{y_c z_c} + A d_y d_z$$

- I_{yy} and I_{zz} are always positive and minimum about the axis passing through the centroid of the body.
- I_{yz} can be positive or negative.
- If either y or z axis is an axis of symmetry then I_{yz} will be zero.

Coordinate Transformation

Definition 1 The coordinate system in which the cross moment of inertia is zero is called the principal coordinate system.

Definition 2 The moment of inertias in the principal coordinate system are called principal moment of inertias.



$$n = y\cos\theta + z\sin\theta$$
$$t = -y\sin\theta + z\cos\theta$$

$$I_{nn} = \int_A t^2 dA = I_{yy}\cos^2\theta + I_{zz}\sin^2\theta - 2I_{yz}\cos\theta\sin\theta$$

$$I_{tt} = \int_A n^2 dA = I_{yy}\sin^2\theta + I_{zz}\cos^2\theta + 2I_{yz}\cos\theta\sin\theta$$

$$I_{nt} = \int_A nt\,dA = (I_{yy} - I_{zz})\cos\theta\sin\theta + I_{yz}(\cos^2\theta - \sin^2\theta)$$

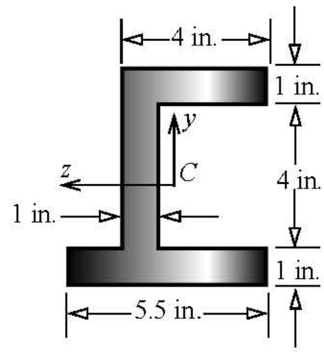
$$\tan 2\theta_p = \frac{-2I_{yz}}{(I_{yy} - I_{zz})} \qquad I_{1,2} = \frac{(I_{yy} + I_{zz})}{2} \pm \sqrt{\left(\frac{I_{yy} - I_{zz}}{2}\right)^2 + I_{yz}^2}$$

- Area moment of intertias are second order tensors.

$$[I] = \begin{bmatrix} I_{yy} & -I_{yz} \\ -I_{zy} & I_{zz} \end{bmatrix} \qquad I_{nn} = \{n\}^T[I]\{n\} \qquad I_{tt} = \{t\}^T[I]\{t\} \qquad -I_{nt} = \{n\}^T[I]\{t\}$$

- Principal area moment of inertias are the eigenvalues of the $[I]$ matrix.
- Buckling occurs about the axis of minimum area moment of inertias.

C6.1 (a) Calculate the principal area moment of inertias for the cross section shown. (b) Determine the axis direction about which buckling would occur.

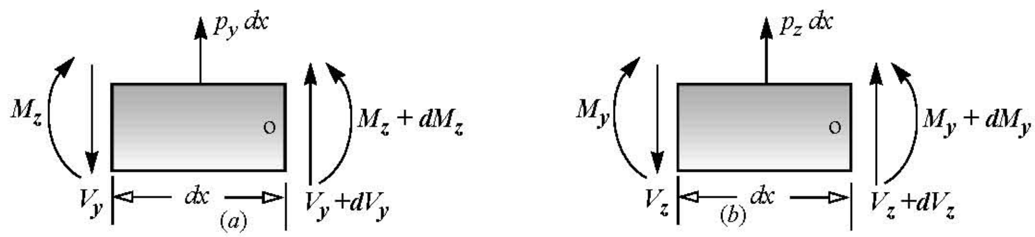


Neutral Axis ($\sigma_{xx} = 0$)

N.A. equation: $y = (\tan\beta)z$ $\qquad \tan\beta = \dfrac{I_{zz} - I_{yz}(M_z/M_y)}{I_{yz} - I_{yy}(M_z/M_y)}$

- The orientation of the neutral axis depends upon the shape of cross-section as well as the external loading.
- Bending normal stress σ_{xx} is maximum at the point which is the farthest from the neutral axis.
- The displacement of the beam is always perpendicular to the neutral axis.

Equilibrium equations.



$$\frac{\mathrm{d}V_y}{\mathrm{d}x} = -p_y \qquad \frac{\mathrm{d}V_z}{\mathrm{d}x} = -p_z$$

$$\frac{\mathrm{d}M_z}{\mathrm{d}x} = -V_y \qquad \frac{\mathrm{d}M_y}{\mathrm{d}x} = -V_z$$

Boundary Value Problem

$$\boxed{V_y = -\frac{\mathrm{d}}{\mathrm{d}x}\left[EI_{zz}\frac{d^2 \mathrm{v}}{dx^2} + EI_{yz}\frac{d^2 w}{dx^2}\right] \qquad V_z = -\frac{\mathrm{d}}{\mathrm{d}x}\left[EI_{yz}\frac{d^2 \mathrm{v}}{dx^2} + EI_{yy}\frac{d^2 w}{dx^2}\right]}$$

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[EI_{zz}\frac{d^2 \mathrm{v}}{dx^2} + EI_{yz}\frac{d^2 w}{dx^2}\right] = p_y \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[EI_{yz}\frac{d^2 \mathrm{v}}{dx^2} + EI_{yy}\frac{d^2 w}{dx^2}\right] = p_z}$$

Assumption 9 The beam is not tapered.

$$EI_{zz}\frac{d^3 \mathrm{v}}{dx^3} + EI_{yz}\frac{d^3 w}{dx^3} = -V_y \qquad EI_{yz}\frac{d^3 \mathrm{v}}{dx^3} + EI_{yy}\frac{d^3 w}{dx^3} = -V_z \tag{6.1a}$$

$$EI_{zz}\frac{d^4 \mathrm{v}}{dx^4} + EI_{yz}\frac{d^4 w}{dx^4} = p_y \qquad EI_{yz}\frac{d^4 \mathrm{v}}{dx^4} + EI_{yy}\frac{d^4 w}{dx^4} = p_z \tag{6.1b}$$

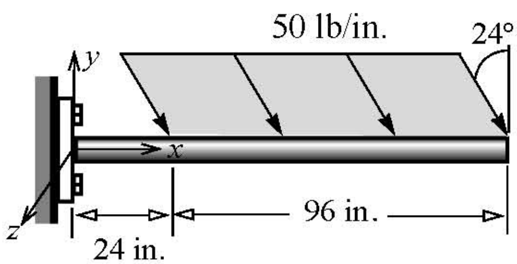
$$\boxed{\frac{d^3 \mathrm{v}}{dx^3} = -\left[\frac{1}{E}\left(\frac{I_{yy}V_y - I_{yz}V_z}{I_{yy}I_{zz} - I_{yz}^2}\right)\right] \qquad \frac{d^3 w}{dx^3} = -\left[\frac{1}{E}\left(\frac{I_{zz}V_z - I_{yz}V_y}{I_{yy}I_{zz} - I_{yz}^2}\right)\right]} \tag{6.2}$$

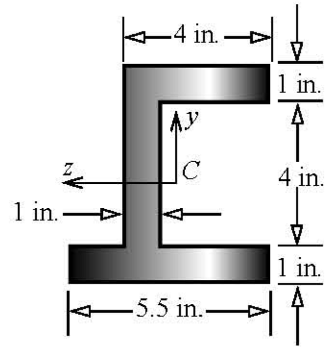
$$\boxed{\frac{d^4 \mathrm{v}}{dx^4} = \frac{1}{E}\left(\frac{I_{yy}p_y - I_{yz}p_z}{I_{yy}I_{zz} - I_{yz}^2}\right) \qquad \frac{d^4 w}{dx^4} = \frac{1}{E}\left(\frac{I_{zz}p_z - I_{yz}p_y}{I_{yy}I_{zz} - I_{yz}^2}\right)} \tag{6.3}$$

Boundary conditions: at each end specify

$$\boxed{\begin{array}{ccc} \mathrm{v} & \text{or} & V_y \\ & \text{and} & \\ \frac{d\mathrm{v}}{dx} & \text{or} & M_z \end{array}} \quad \text{and} \quad \boxed{\begin{array}{ccc} w & \text{or} & V_z \\ & \text{and} & \\ \frac{dw}{dx} & \text{or} & M_y \end{array}}$$

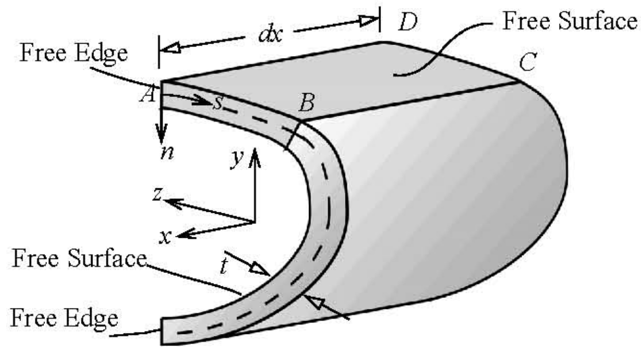
C6.2 A cantilever beam is loaded such that there is no twist. The distributed load acts in the *y*-*z* plane at an angle of 24^0 from the *x*-*y* plane as shown below. On a section at $x = 60$ in, determine: (a) the orientation of the neutral axis. (b) the maximum bending normal stress in the section.

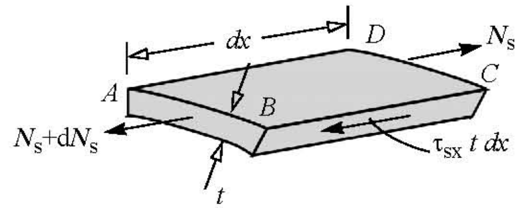




C6.3 The modulus of elasticity for the beam in problem C6.2 is E = 30,000ksi. Determine the deflection of the beam at x = 60 inch and show that it is perpendicular to the neutral axis.

Shear stress in thin open sections





Equilibrium Equations: $(N_S + dN_S) - N_S + \tau_{sx} t dx = 0 \qquad \tau_{sx} t = -\dfrac{dN_s}{dx}$

Axial Force: $N_s = \int_{A_s} \sigma_{xx}\ dA \qquad \tau_{sx} t = -\dfrac{d}{dx}\int_{A_s} \sigma_{xx}\ dA$

Definition 3 The direction of the s-coordinate is from the free surface towards the point where shear stress is being calculated.

Definition 4 The area A_s is the area between free edge and the point at which the shear stress is being evaluated.

$$\tau_{sx} t = -\frac{d}{dx}\int_{A_s}\left[-\left(\frac{I_{yy}M_z - I_{yz}M_y}{I_{yy}I_{zz} - I_{yz}^2}\right)y - \left(\frac{I_{zz}M_y - I_{yz}M_z}{I_{yy}I_{zz} - I_{yz}^2}\right)z\right] dA$$

$$\tau_{sx} t = \frac{d}{dx}\left[\left(\frac{I_{yy}M_z - I_{yz}M_y}{I_{yy}I_{zz} - I_{yz}^2}\right)\int_{A_s} y dA + \left(\frac{I_{zz}M_y - I_{yz}M_z}{I_{yy}I_{zz} - I_{yz}^2}\right)\int_{A_s} z dA\right]$$

We define the first moment of the area A_s as:

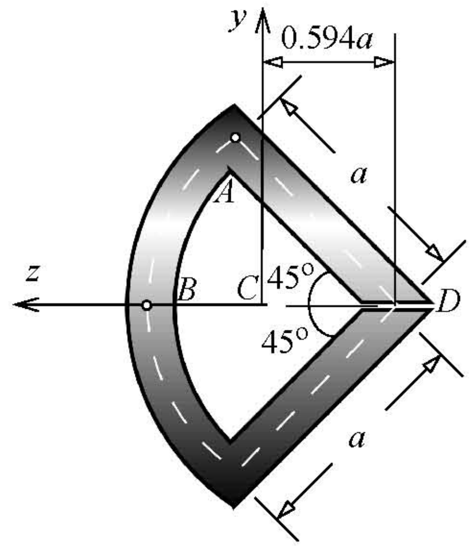
$$Q_z = \int_{A_s} y dA \qquad Q_y = \int_{A_s} z dA$$

Assumption 10 The beam is not tapered. ‘

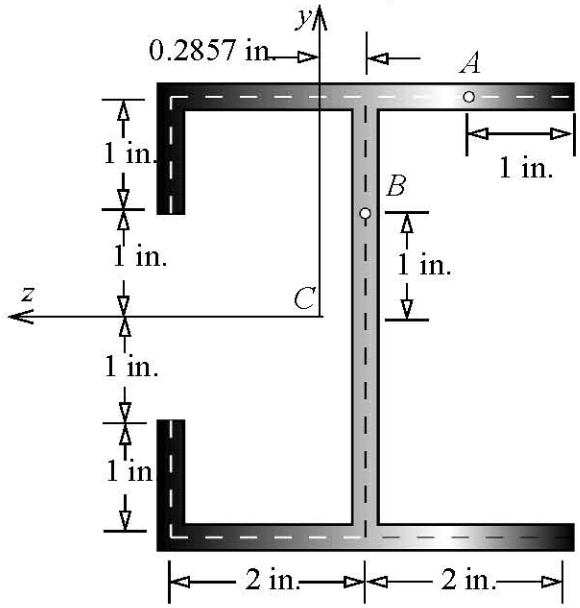
$$\tau_{sx} t = \left[\frac{I_{yy}(dM_z/dx) - I_{yz}(dM_y/dx)}{I_{yy}I_{zz} - I_{yz}^2}\right]Q_z + \left[\frac{I_{zz}(dM_y/dx) - I_{yz}(dM_z/dx)}{I_{yy}I_{zz} - I_{yz}^2}\right]Q_y$$

$$\boxed{q = \tau_{sx} t = -\left[\frac{I_{yy}Q_z - I_{yz}Q_y}{I_{yy}I_{zz} - I_{yz}^2}\right]V_y - \left[\frac{I_{zz}Q_y - I_{yz}Q_z}{I_{yy}I_{zz} - I_{yz}^2}\right]V_z}$$

C6.4 A thin cross-section of uniform thickness t is shown below. If shear stresses were to be found at points A and B what values of Q_y and Q_z are needed for the calculation. Assume $t \ll a$ and gap at D is of negligible thickness. Report the values of Q_y and Q_z in terms of t and a.

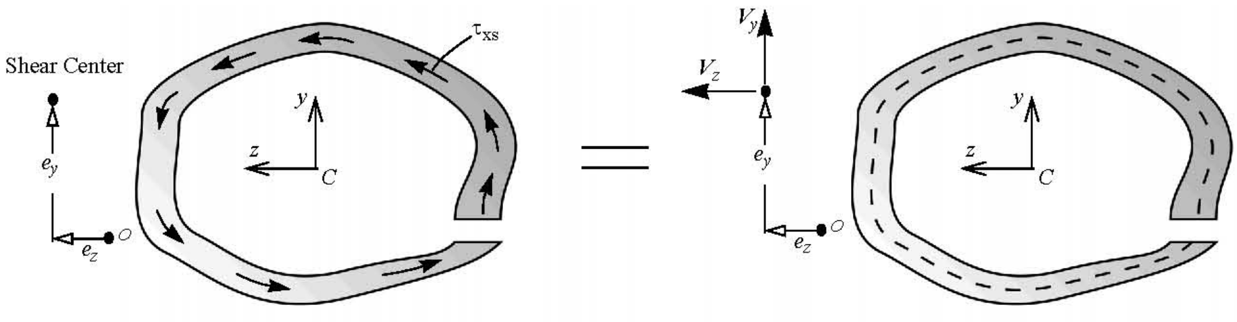


C6.5 Shear forces on the cross-section shown in C6.5 were calculated as $V_y = 10\ kips$ and $V_z = -5\ kips$. The cross section has a uniform thickness of 1/8 *in.* Determine the bending shear stresses at points *A* and *B* and report your answers as τ_{xy} and τ_{xz}.



Shear center

From statics we know that any distributed force can be replaced by a force and a moment at any point, or, by a *single force (and no moment) at a specific point*. The specific point at which the shear stress (shear flow) can be represented by just shear forces V_y and V_z (components of a single force) and no internal torque is called the shear center.



Definition 5 Shear center is a point in space at which the shear stress due to bending can be replaced by statically equivalent internal shear forces and no internal torque.

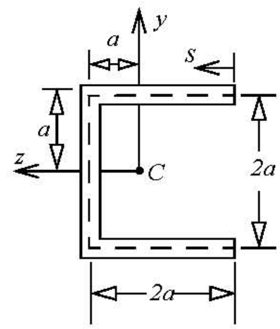
or

Definition 6 Shear center is a point in space such that if the line of action of external forces pass through the point then the cross-section will not twist.

- Each cross-section has a unique shear center associated with it.
- Shear center depends only on the geometry and is independent of the loading.
- Shear center lies on the axis around which the shear stress distribution is symmetric.
- Shear center de-couples the shear stresses due to bending from the shear stresses due to torsion.
- If bending forces are not to produce any axial or torsional deformation then the external forces must be along the line joining the centroid and the shear center of the cross-section.

C6.6 The cross-section shown has a uniform thickness t. Assuming $t \ll a$ the shear stresses in the cross section were found and are as given. (a) Replace the shear stresses by equivalent shear forces and torque acting at the centroid C. (b) Determine the location of the point where the shear stresses can be replaced by just shear forces and no torque, i.e., determine the shear center.

$\tau_{xy} = 0$	$\tau_{xz} = Ks/t$	$0 \le s < 2a$
$\tau_{xy} = -K(-4a^2 + 6as - s^2)/(2at)$	$\tau_{xz} = 0$	$2a < s < 4a$
$\tau_{xy} = 0$	$\tau_{xz} = K(s - 6a)/t$	$4a < s \le 6a$

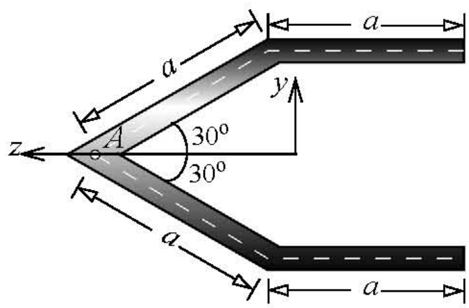


C6.7 The cross-sections shown below has a uniform thickness t. Assume $t \ll a$. Assume a shear force $V_y = V$ acts on the cross section.

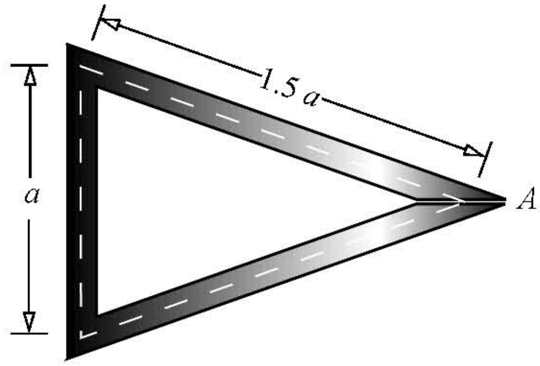
(a) Determine the shear flow on the entire cross section.

(b) Replace the shear flow by equivalent force and moment at point A.

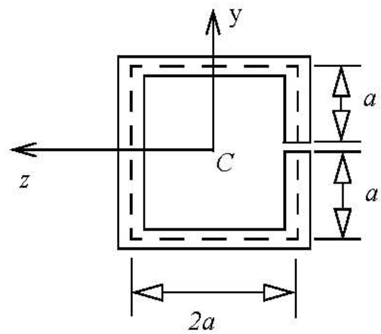
(c) Determine the location of the point where the shear flow can be replaced by just the shear flow with no moment.



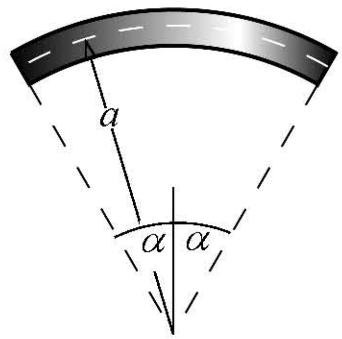
C6.8 The cross-section shown below has a uniform thickness t. Assuming $t \ll a$ determine the location of shear centers with respect to point A.



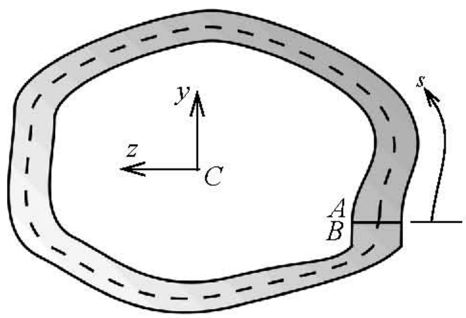
C6.9 A thin walled open cross-section with a uniform thickness 't' is shown. Assume $t \ll a$ and the gap is of negligible thickness. Determine the coordinates of the shear center e_y and e_z with respect to the centroid at C.

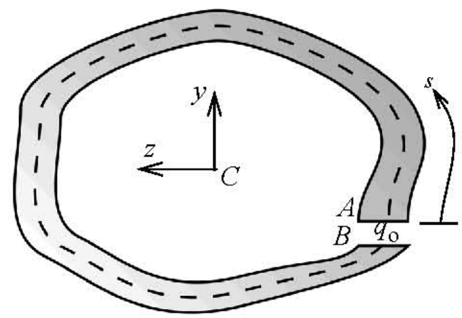


C6.10 The cross-section shown below has a uniform thickness t and boundaries made from circular arcs. Assuming $t \ll a$ determine the location of shear centers with respect to point A in terms of radius a and angle α.



Shear stresses in thin closed sections





$$q_c = q_o + q$$

q_c is the shear flow in the closed section at any point,

q is the shear flow of the open section, and

q_o is the unknown shear flow at the starting point that has to be determined.

Shear strain can be written as:

$$\gamma_{xs} = \frac{\partial u}{\partial s} + \frac{\partial v_s}{\partial x} = \frac{\tau_{xs}}{G}$$

u and v_s are displacement in the x and s direction, respectively, and

G is the shear modulus of elasticity.

$$\int_{s_A}^{s_B} \frac{\partial u}{\partial s} ds = \oint \left[\frac{\tau_{xs}}{G} - \frac{\partial v_s}{\partial x}\right] ds \quad \text{or} \quad u(s_B) - u(s_A) = \oint \left[\frac{\tau_{xs}}{G} - \frac{\partial v_s}{\partial x}\right] ds$$

Assumption 2a through Assumption 2c implies: Cross-section shape and dimension undergoes negligible change. This implies that no point on the cross-section moves relative to the other in the s-direction i.e., $v_s = 0$ in pure bending.

Noting that $u(s_B) = u(s_A)$ we obtain:

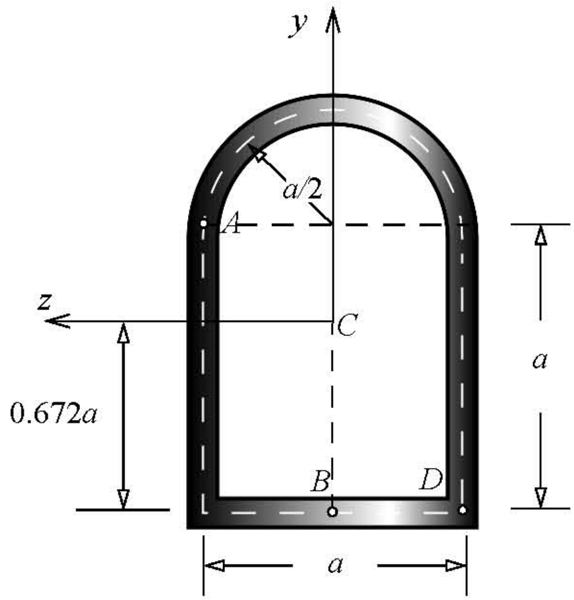
$$\oint\left(\frac{q_c}{t}\right) ds = \oint\left(\frac{q_o + q}{t}\right) ds = 0$$

If the thickness is uniform across the cross-section.

$$q_o = -\frac{1}{S}\oint q\, ds$$

where, S is the total path length of the perimeter of the cross-section.

C6.11 The thin cross section shown below has a uniform thickness t and is subjected to a shear force $V_y = V$ acting through the shear center. Determine the shear stress at points A and B in terms of V, a, and t.



C6.12 The thin cross section shown above has a uniform thickness t and is subjected to a shear force $V_z = V$ acting through the shear center. Starting with point D, determine the shear stress at points A and B in terms of V, a, and t.

C6.13 Determine the shear center of the cross-section shown above relative to centroid C.

C6.14 A cantilever beam is loaded as shown below. The cross-section has a uniform thickness of t = 1/4 *in*. Determine the normal and shear stress at points *A* and *B* in cartesian coordinates on a section next to the wall.